Math. 642 Test 1: Take-home part. This take-home part of Test 1 is due Wednesday, 3/16/2016, at 4 pm. You may consult any written or online source. You may also consult your instructor; however, you may *not* consult anyone else. (The part of the exam is worth 120 points.)

1. Let $J[y] = \int_0^1 (y^{(k+1)}(x))^2 dx$. Let the admissible set for J be all piecewise C^{k+1} curves for which $y(j/n) = y_j$, $j = 0, \ldots n$, with the discontinuities in $y^{(k+1)}$ occurring only at the points $x_j = j/n$. In addition, assume that $y^{(\ell)}(0)$ and $y^{(\ell)}(1)$ are given for $\ell = 0, \ldots, k$.

   (a) **(20 pts.)** Let $f(x)$ be continuous on $[a, b]$ and suppose that, for all $\eta \in C^{k+1}[a, b]$ satisfying $\eta^{(j)}(a) = \eta^{(j)}(b) = 0$, $j = 0, \ldots, k$, we have $\int_a^b f(x)\eta^{(k+1)}(x)dx = 0$. Show that $f(x)$ is a polynomial of degree k.

   (b) **(10 pts.)** Use the previous part to show that the minimizer $y(x)$ for J is in the finite element space $S^{1/n}(2k+1, 2k)$. (Hint: Show that $y^{(m)}(x_j^-) = y^{(m)}(x_j^+)$ for $m = k+1, \ldots, 2k$.)

2. Let $w = f(z)$ be analytic in a region containing the disk $|z| \le 1$, and suppose that $f(0) = 0$, $f'(0) \neq 0$. In addition, suppose that $f(z)$ maps this disk one-to-one and onto a region in the w plane containing a disk $|w| \le a$.

   (a) **(10 pts.)** Show that the function inverse to f, $g(w)$, is given by the contour integral

   $$g(w) = \frac{1}{2\pi i} \oint_{|\zeta|=1} \frac{\zeta f'(\zeta)}{f(\zeta) - w} d\zeta. \tag{2}$$

   (b) **(10 pts.)** For $f(z) = (z-2)^2 - 4$ and $|w|$ small, expand the integrand in (2) in a power series in w. Calculate the coefficients in this series and verify that the result is $z = 2 - \sqrt{w+4}$, where the square root uses principal branch in which $\arg(z) \in (-\pi, \pi]$.

3. A mass m is attached to a pendulum of length ℓ and negligible weight. The pendulum itself is attached to a fixed pivot and allowed to swing freely, with the mass subject only to gravity. Take the pivot to be the origin.

(a) **(5 pts.)** Using spherical coordinates, where θ is the colatitude and ϕ is the longitude, find the Hamiltonian for the system.

(b) **(10 pts.)** Use part (b) above to find two constants of the motion. Use these in conjunction with the Hamiltonian from part (a) to find a first order nonlinear differential equation for θ. (You don't need to solve the equation.)

(c) **(10 pts.)** If $\frac{d\phi}{dt}(0) = 0$, show that $\phi(t) = \phi(0)$, and that the system reduces to the simple pendulum in the plane $\phi = \phi(0)$.

4. The following is a special case of the Paley-Wiener Theorem. An entire function $f(z)$ – i.e., analytic in $\mathbb{C}$ – is said to be of *exponential type* if there exist constants $A > 0$ and $\sigma > 0$ such that $|f(z)| \le Ae^{\sigma|z|}$ for all $z \in \mathbb{C}$.

   (a) **(20 pts.)** Prove this: *If f is of exponential type and f is uniformly bounded on the real axis, then there exists a constant $M > 0$ such that $|f(z)| \le Me^{\sigma|y|}$, where $y = \mathrm{Im}(z)$.* (Hint: Using appropriate rectangles, apply the maximum principle theorem to two different functions: $f(z)e^{-\sigma z}$, for $x \ge 0$, and, for $x \le 0$, $f(z)e^{\sigma z}$.) Note: You may *not* use problem 6.2.13 in the text.

   (b) **(10 pts.)** Prove this: *Let f be of exponential type and suppose that $f(x) \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. Then, for all $\omega \in \mathbb{R}$ for which $|\omega| > \sigma$, we have*
$$\int_{-\infty}^{\infty} f(x)e^{i\omega x}dx = 0.$$

5. **(15 pts.)** Recall that $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$, which is valid when z is in the right half plane, $\Re(z) > 0$. Apply the dominated convergence theorem to the difference quotient $(\Gamma(z+h) - \Gamma(z))/h$ to show that $\Gamma'(z) = \int_0^\infty t^{z-1}\ln(t)e^{-t}dt$.