Final Examination

This take-home exam is due at 4 pm on Wednesday, May 8. You may consult any written or online source. You may *not* consult any person, either a fellow student or faculty member, except your instructor

1. **(15 pts.)** Let T be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. In addition, suppose that 0 is in the continuous spectrum of T. If $\mathcal{R} = \text{Range}(T)$, show that $L = T^{-1}$, with domain $\mathcal{D} = \mathcal{R}$, is a self-adjoint operator.

2. Consider the self adjoint operator $Lu = -u''$, $0 < x < \infty$, with $D_L = \{u \in L^2([0,\infty)) \colon u'' \in L^2([0,\infty)),\ u'(0) = 0\}$.

   (a) **(10 pts.)** Find the Green's function for L.

   (b) **(15 pts.)** Use Stone's formula to find the associated spectral transform.

3. **(20 pts.)** Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let L be a self-adjoint operator with domain $\mathcal{D}$. Show that if $c := \inf_{f\in\mathcal{D},\, \|f\|=1} \langle Lf, f\rangle$, $-\infty < c < \infty$, then $c \in \sigma(L)$ and $(-\infty, c) \subset \rho(L)$. Equivalently, $c = \min \sigma(L)$.

4. Suppose f is a band-limited signal, with $\widehat{f}(\omega) = 0$ for $|\omega| \ge \Omega$. In addition, for $a > 1$, let $g_a = \mathcal{F}^{-1}(\widehat{g}_a)$, where $\widehat{g}_a$ is the function whose graph is given by Figure 1. In addition, let f be sampled at the rate $a\Omega/\pi$ which is higher that the Nyquist rate, Ω/π. This means that we are oversampling f.

   (a) **(5 pts.)** Show that, on $|\omega| \le a\Omega$, $\widehat{f}(\omega)$, which is 0 for $|\omega| \ge \Omega$, can be represented by the $2a\Omega$ periodic Fourier series,

   $$\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-in\pi\omega/a\Omega} \quad \text{with } c_{-n} = \frac{\pi}{a\Omega} f\Big(\frac{n\pi}{a\Omega}\Big), \tag{1}$$

   In addition, $\widehat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_{-n} e^{-in\pi\omega/a\Omega}\, \widehat{g}_a(\omega)$, on $[-a\Omega, a\Omega]$, because $\widehat{g}_a(\omega) = 1$ on $[-\Omega, \Omega]$ and $\widehat{f}(\omega) = 0$ on $|\omega| \ge \Omega$.

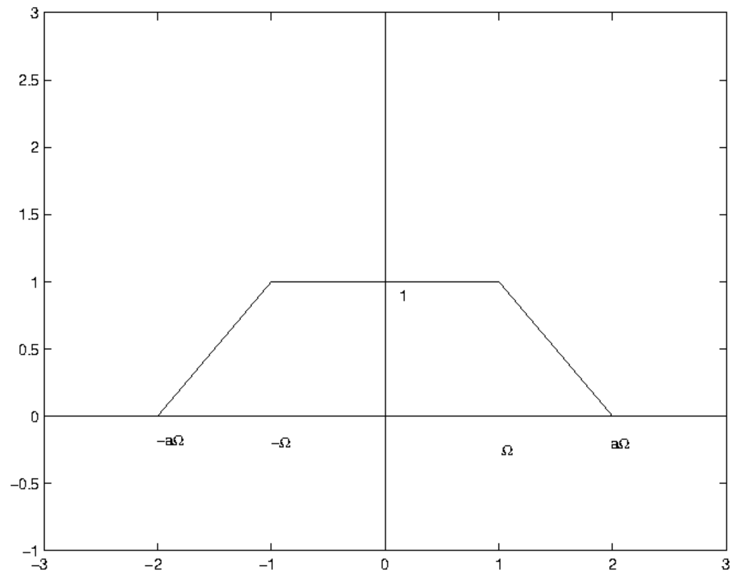

Figure 1: Graph of $\widehat{g}_a$

(b) **(5 pts.)** Show that

$$g_a(t) = \frac{\cos(\Omega t) - \cos(a\Omega t)}{\pi(a-1)\Omega t^2} \tag{2}$$

(c) **(5 pts.)** Use the previous two parts to show that

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}) g_a(t - \frac{n\pi}{a\Omega}). \tag{3}$$

(d) **(10 pts.)** Applying Parseval's theorem to the Fourier series (1), show that $\sum_{n\in\mathbb{Z}} |f(\frac{n\pi}{a\Omega})|^2 = \frac{a\Omega}{2\pi^2}\|f\|^2_{L^2(\mathbb{R})}$. For all $|t| \le \frac{N\pi}{2a\Omega}$, show that $|f(t) - \sum_{|n|\le N-1} \frac{\pi}{a\Omega} f(\frac{n\pi}{a\Omega}) g_a(t - \frac{n\pi}{a\Omega})| \le C\|f\|_{L^2(\mathbb{R})} N^{-3/2}$. What is the rate for the standard sampling series? Does oversampling improve the rate of convergence?

5. **(15 pts.)** Consider the sum $S(n) = \sum_{k=0}^{n} k!\binom{n}{k} n^{-k}$. Show that for large n we have that $S(n) \sim \sqrt{\frac{n\pi}{2}}$. Hint: $k!n^{-k-1} = \int_0^\infty e^{-nx} x^k dx$.