Midterm

Take-home part. This take-home part of the midterm is due Friday, 10/19/2012. You may consult any written or online source. You may *not* consult anyone, except your instructor

1. **(25 pts.)** Prove this: Let $m \geq 0$. If f is 2π-periodic and $f^{(m)}$ is piecewise smooth and c_r is the r^{th} Fourier coefficient for f, then, for all $r \neq 0$,
$$|c_r| \leq C|r|^{-(m+1)}, \tag{1}$$
where C is a constant independent of r.

2. Let $f(x)$ be a continuous 2π-period function, with its N^{th} partial sum being $S_N(x) = \sum_{\ell=-N}^{N} c_\ell e^{i\ell x}$. Finally, let $\mathcal{F}_n$ be the discrete Fourier transform on $\mathcal{S}_n$. In addition, for a given function g let
$$\mathcal{F}_n[g]_k = \sum_{j=0}^{n-1} g\left(\frac{2j\pi}{n}\right)\bar{w}^{jk}.$$
   (a) **(5 pts.)** Show that $\frac{1}{n}\mathcal{F}_n[S_N]_k = c_k$, provided $N \leq (n-1)/2$.
   (b) **(10 pts.)** Show that $\left|c_k - \frac{1}{n}\mathcal{F}_n[f]_k\right| \leq \|S_{\lfloor (n-1)/2 \rfloor} - f\|_\infty$, if $|k| \leq (n-1)/2$.
   (c) **(10 pts.)** Use part 2b above and equation (1) to show that if f is 2π-periodic and $f^{(m)}$ is piecewise smooth, then there is a constant C that is independent of k and n such that
$$\left|c_k - \frac{1}{n}\mathcal{F}_n[f]_k\right| \leq Cn^{-m}, \text{ for } |k| \leq (n-1)/2.$$

3. **(25 pts.)** Prove this version of the sampling theorem: *Let* $\Omega > 0$, $\lambda > 1$, *and suppose that* $f, g \in L^2$ *are band-limited, with* $\operatorname{supp} \hat{f} \subseteq [-\Omega, \Omega]$ *and* $\operatorname{supp} \hat{g} \subseteq [-\lambda\Omega, \lambda\Omega]$. *If* $\hat{g}(\omega) \in C(\mathbb{R})$ *satisfies* $\hat{g}(\omega) = 1$ *on* $[-\Omega, \Omega]$, *then*
$$f(t) = \frac{\pi}{\lambda\Omega} \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\lambda\Omega}\right) g\left(t - \frac{n\pi}{\lambda\Omega}\right)$$

4. **(25 pts.)** Consider the inner product for the Sobolev space $H^1(\mathbb{R})$,
$$\langle f, g\rangle_{H^1} := \int_{\mathbb{R}} (f'\bar{g}' + f\bar{g})dx.$$
where f, g, f', g' are all in $L^2(\mathbb{R})$. Show that $H^1(\mathbb{R}) \subset C_0(\mathbb{R})$, and that if $\kappa(x) = e^{-|x|}/2$, then $f(x) = \langle f, \kappa(x - \cdot)\rangle_{H^1}$.