Change of Basis

Coordinate vectors. This are notes covering changing bases/coordinates. Here is the setup for all of the problems. We begin with a vector space V that has an ordered basis $E = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. (We always keep the same order for vectors in the basis.) By the "coordinate theorem," if $\mathbf{v} \in V$, then we can always express $\mathbf{v} \in V$ in one and only one way as a linear combination of the the vectors in E. Specifically, for any $\mathbf{v} \in V$ there are scalars $x_1, \ldots, x_n$ such that

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n\,. \tag{1}$$

Moreover, by the same theorem, we have an isomorphism between V and $\mathbb{R}^n$:

$$[\mathbf{v}]_E = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

The column vector $[\mathbf{v}]_E$ is the coordinate vector of $\mathbf{v}$ relative to E, and the x_j's are the coordinates of $\mathbf{v}$. The isomorphism just means that these properties hold:

$$[\mathbf{v} + \mathbf{w}]_E = [\mathbf{v}]_E + [\mathbf{w}]_E \quad \text{and} \quad [c\mathbf{v}]_E = c[\mathbf{v}]_E. \tag{2}$$

Examples. Let $V = \mathcal{P}_3$ and $E = \{1, x, x^2\}$. What is the coordinate vector $[5 + 3x - x^2]_B$? Answer:

$$[5 + 3x - x^2]_E = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}.$$

If we ask the same question for $[5 - x^2 + 3x]_E$, the answer is the *same*, because to find the coordinate vector we have to *order* the basis elements so that they are in the same order as E.

Let's turn the question around. Suppose that we are given

$$[p]_E = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},$$

then what is p? Answer: $p(x) = 3 \cdot 1 + 0 \cdot x + (-4) \cdot x^2 = 3 - 4x^2$.

Let's try another space. Let $V = \text{span}\{e^x, e^{-x}\}$, which is a subspace of $C(-\infty, \infty)$. Here, we will take $E = \{e^x, e^{-x}\}$. What are coordinate vectors for $\sinh(x)$ and $\cosh(x)$? Solution: Since $\sinh(x) = \frac{1}{2}e^t - \frac{1}{2}e^{-x}$ and $\cosh(x) = \frac{1}{2}e^x + \frac{1}{2}e^{-x}$, these vectors are

$$[\sinh(x)]_E = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \quad \text{and} \quad [\cosh(x)]_E = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}. \tag{3}$$

Change of basis Suppose that $F = [\mathbf{w}_1, \ldots, \mathbf{w}_n]$ is a second basis for V. How do we relate coordinates relative to E to those of F? How do we change from one set of coordinates to another?

Suppose that we know the E-coordinate vector of $\mathbf{v}$, $\mathbf{x} := [\mathbf{v}]_E$ and we want F-coordinate vector $\mathbf{y} := [\mathbf{v}]_F$. To do this, we first find the F-coordinates of the E-basis vectors; these are just the column vectors below.

$$\mathbf{s}_1 = [\mathbf{v}_1]_F,\ \mathbf{s}_2 = [\mathbf{v}_2]_F,\ \ldots,\ \mathbf{s}_n = [\mathbf{v}_n]_F\,.$$

Now, in the representation (1) for $\mathbf{v}$ in the E-basis, take the F-coordinates of both sides to get

$$\mathbf{y} = [\mathbf{v}]_F = [x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n]_F.$$

Using the properties of coordinate vectors in (2), we see that

$$\begin{aligned} \mathbf{y} = [\mathbf{v}]_F &= x_1[\mathbf{v}_1]_F + x_2[\mathbf{v}_2]_F + \cdots + x_n[\mathbf{v}_n]_F \\ &= x_1\mathbf{s}_1 + x_2\mathbf{s}_2 + \cdots + x_n\mathbf{s}_n \\ &= \underbrace{[\mathbf{s}_1\ \mathbf{s}_2 \cdots \mathbf{s}_n]}_{S}\,\mathbf{x} = S\mathbf{x} \end{aligned}$$

The matrix S is the transition matrix from E-coordinates to F-coordinates. When we want to emphasize this, we will write $S_{E\to F}$, instead of just S.

Examples. Start out with $V = P_3$. Let $E = [x+1, x-1, 1+x+x^2]$ and let $F = [1, x, x^2]$. To find the change of basis matrix $S_{E\to F}$, we need the F coordinate vectors for the E basis. These are easy to find.

$$\mathbf{s}_1 = [1{+}x]_F = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{s}_2 = [x{-}1]_F = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \mathbf{s}_3 = [1{+}x{+}x^2]_F = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

By what we said above, the transition matrix is

$$S_{E\to F} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

To get the transition matrix in the reverse direction, $F \to E$, we just need to calculate $S_{F\to E} = S_{E\to F}^{-1}$. In this example, the answer is

$$S_{F\to E} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To clarify this, start with $[p]_E = \begin{pmatrix} 1 & -2 & 5 \end{pmatrix}^T$ – so that $p(x) = 1 \cdot (1+x) + (-2)\cdot(x-1) + 5\cdot(1+x+x^2)$. Then $[p]_F = S_{E\to f}[p]_E = \begin{pmatrix} 8 & 4 & 5 \end{pmatrix}^T$, or $p(x) = 8\cdot 1 + 4\cdot x + 5\cdot x^2$.

Let's look at a second example. Let $V = \mathrm{span}\{\mathrm{e^x}, \mathrm{e^{-x}}\}$. The set $E = [e^x, e^{-x}]$ is linearly independent, and therefor a basis. A second basis is $F = [\cosh(x), \sinh(x)]$. By definition of cosh and sinh, we have $\cosh(x) = \frac{1}{2}(e^x + e^{-x}$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$. However, this is a problem, because what we need is the E-basis vectors in terms of F-coordinates. What we *have* is F-basis vectors in E-coordinates. To deal with this situation, which is pretty common, we first find $S_{F\to E}$ and then invert to get $S_{E\to F}$. Using $[\cosh(x)]_E$ and $[\sinh(x)]_E$ from (3), we have

$$S_{F\to E} = \big[[\cosh(x)]_E\ [\sinh(x)]_E\big] = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}.$$

Finding the inverse of this matrix gives us the transition matrix that we wanted in the first place:

$$S_{E\to F} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Here is our final example $V = P_3$, $E = [1, x, x^2]$ and $F = [1, x, \frac{1}{2}(3x^2-1)]$. Again we want $S_{E\to F}$. However, we are in the same situation as the last problem. We know the F-basis in E-coordinates, not the E-basis in F-coordinates. As before, we first find

$$S_{F\to E} = \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix}. \tag{4}$$

We then find the inverse of this matrix to get the transition matrix from E to F:

$$S_{E\to F} = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} \tag{5}$$

Change of basis for matrices for linear transformations. The matrix A that represents a linear transformation $L : V \to V$ relative to a basis $E = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is

$$A = \big[\, [L(\mathbf{v}_1)]_E \ [L(\mathbf{v}_2)]_E \ \cdots [L(\mathbf{v}_n)]_E \big].$$

The equation $L(\mathbf{v}) = \mathbf{w}$ is completely equivalent to the matrix equation $A[\mathbf{v}]_E = [\mathbf{w}]_E$. Of course, we could have used another basis $F = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]$, with another matrix

$$B = \big[\, [L(\mathbf{w}_1)]_F \ [L(\mathbf{w}_2)]_F \ \cdots [L(\mathbf{w}_n)]_F \big]$$

representing L. If we know A and want to find B, start with $A[\mathbf{v}]_E = [\mathbf{w}]_E$. Use $[\mathbf{v}]_E = S_{FtoE}[\mathbf{v}]_F$ to get $AS_{F\to E}[\mathbf{v}]_F = [\mathbf{w}]_E$. Then, note that $[\mathbf{w}]_F = S_{E\to F}[\mathbf{w}]_E$. Combining this with the previous equation gives $S_{F\to E}AS_{F\to E}[\mathbf{v}]_F = [\mathbf{w}]_F$. Letting $S = S_{F\to E}$, we obtain this relation between A and B:

$$B = S^{-1}AS$$

An example. Let $V = P_3$ and take $E = [1, x, x^2]$ and $F = [1, x, \frac{1}{2}(3x^2 - 1)]$. The linear transformation for this example is $L(p) = ((x^2 - 1)p')'$. To find the matrix A that represents L, we first apply L to each of the basis vectors in B.

$$L(1) = 0,\ L(x) = 2x,\ \text{and}\ L(x^2) = -2 + 6x^2.$$

Next, we find the E-basis coordinate vectors for each of these.

$$[0]_E = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad [2x]_E = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \quad [-2 + 6x^2]_E = \begin{pmatrix} -2 \\ 0 \\ 6 \end{pmatrix},$$

and so them matrix that represents L is

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

We have already calculated the transition matrices, which are given in (4) and (5). Thus, since $B = S_{E\to F}AS_{S\to E}$, we have

$$B = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 2/3 \end{pmatrix} \begin{pmatrix} 0 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$