Coordinate Vectors and Examples

Francis J. Narcowich
Department of Mathematics
Texas A&M University

June 2013

Coordinate vectors. This is a brief discussion of coordinate vectors and the notation for them that I presented in class. Here is the setup for all of the problems. We begin with a vector space V that has a basis $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ – i.e., a spanning set that is linearly independent. We always keep the same order for vectors in the basis. Technically, this is called an *ordered* basis. The following theorem, Theorem 3.2, p. 139, in the text gives the necessary ingredients for making coordinates:

Theorem 1 (Coordinate Theorem) *Let $V = \mathrm{Span}(B)$, where the set $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. Then, every $\mathbf{v} \in V$ can represented in exactly one way as linear combination of the $\mathbf{v}_j$'s if and only if $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent – hence, B is a basis, since it spans. In particular, if B is a basis, there are unique scalars $x_1, \ldots, x_n$ such that*

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n\,. \tag{1}$$

This theorem allows us to assign coordinates to vectors, provided we don't change the order of the vectors in B. That is, B is is an *ordered* basis. When order matters we write $B = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$. (For example, for $\mathcal{P}_3$, the ordered basis $[1, x, x^2]$ is different than $[x, x^2, 1]$.) If the basis is ordered, then the coefficient x_j in equation (1) corresponds to $\mathbf{v}_j$, and we say that the x_j's are the coordinates of $\mathbf{v}$ relative to B. We collect them into the *coordinate vector*

$$[\mathbf{v}]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

Isomorphisms. The correspondence between a vector $\mathbf{v}$ in V and $[\mathbf{v}]_B$, its coordinate vector $[\mathbf{v}]_B$ in $\mathbb{R}^n$, has some nice properties. First, the correspondence is *one-to-one* and *onto*. This means that for every $\mathbf{v}$ there is exactly one column vector $[\mathbf{v}]_B$, and, conversely, every column vector $(x_1\ x_2 \cdots x_n)^T$ in $\mathbb{R}^n$ corresponds to $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n$. Second, the correspondence preserves vector addition and multiplication by a scalar: If $\mathbf{v} \leftrightarrow \mathbf{x} := [\mathbf{v}]_B$ and $\mathbf{w} \leftrightarrow \mathbf{y} := [\mathbf{w}]_B$, then

$$\mathbf{v} + \mathbf{w} \leftrightarrow \mathbf{x} + \mathbf{y} \text{ and } c\mathbf{v} \leftrightarrow c\mathbf{x}.$$

A one-to-one, onto correspondence between vector spaces that preserves the operations of addition and multiplication by a scalar is called an *isomorphism*. As far as vector operations go, two spaces that are isomorphic are equivalent. We can put this another way: $[\mathbf{v}]_B + [\mathbf{w}]_B = [\mathbf{v} + \mathbf{w}]_B$ and $[c\mathbf{v}]_B = c[\mathbf{v}]_B$. In terms of a linear combinations, we have $[a\mathbf{v} + b\mathbf{w}]_B = a[\mathbf{v}]_B + b[\mathbf{w}]_B$.

Examples. Here are some examples. Let $V = \mathcal{P}_2$ and $B = \{1, x, x^2\}$. What is the coordinate vector $[5 + 3x - x^2]_B$? Answer:

$$[5 + 3x - x^2]_B = \begin{pmatrix} 5 \\ 3 \\ -1 \end{pmatrix}.$$

If we ask the same question for $[5 - x^2 + 3x]_B$, the answer is the *same*, because to find the coordinate vector we have to *order* the basis elements so that they are in the same order as B.

Let's turn the question around. Suppose that we are given

$$[p]_B = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix},$$

then what is p? Answer: $p(x) = 3 \cdot 1 + 0 \cdot x + (-4) \cdot x^2 = 3 - 4x^2$.

We want to illustrate the isomorphism between $\mathcal{P}_2$ and $\mathbb{R}^3$. For example, if $p(x) = 1 - 4x + 5x^2$ and $q(x) = -4 + x + x^2$, the the polynomial $p + q$ is $p(x) + q(x) = -3 - 3x + 6x^2$. The coordinate vectors for these in $\mathbb{R}^3$ are

$$[p]_B = \begin{pmatrix} 1 \\ -4 \\ 5 \end{pmatrix},\ [q]_B = \begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix},\ [p+q]_B = \begin{pmatrix} -3 \\ -3 \\ 6 \end{pmatrix} = [p]_B + [q]_B.$$

The last equation is exactly what we mean by "preserving vector addition." Similar examples hold for multiplication by a scalar.

We now want to use this isomorphism to determine whether or not a set of polynomials are linearly dependent or linearly independent. Let $S = \{x+1, x^2, x^2+x+1\}$. We need to look at the homogeneous equation

$$c_1(x+1) + c_2x^2 + c_3(x^2+x+1) \equiv 0.$$

Because linear combinations of vectors are equivalent to linear combinations of coordinate vectors, we have

$$c_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

We may put this system in augmented form and row reduce it:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

This system has the solution $c_1 = -t$, $c_2 = -t$, and $c_3 = t$. If we take $t = 1$, then $c_1 = -1$, $c_2 = -1$, and $c_3 = 1$. Thus, the original homogeneous system involving polynomials has a nonzero solution and so S is linearly dependent.

Let's try another space. Let $V = \{y \in C^{(2)} \,|\, y'' - y = 0\}$. This is the space of solutions to $y'' - y = 0$. This ODE has solutions that are linear combinations of two linearly independent solutions, e^t, e^{-t}. Thus $B = \{e^t, e^{-t}\}$ is a basis for V. It follows that every solution $y = c_1e^t + c_2e^{-t}$ has the coordinate representation

$$[y]_B = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

To be more definite, if y is a solution that satisfies $y(0) = 3$ and $y'(0) = 1$, then $y(x) = 2e^t + e^{-t}$, which has the coordinate vector

$$[2e^t + e^{-t}]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Suppose that we have a synthesizer that can produce two frequencies, ν_1 and ν_2. The sounds that it can produce are superpositions of sines and cosines with these frequencies; they are thus in the vector space

$$V = \mathrm{Span}\{\cos(2\pi\nu_1 t), \sin(2\pi\nu_1 t), \cos(2\pi\nu_2 t), \sin(2\pi\nu_2 t)\}.$$

It's not hard to show that the four functions involved are linearly independent and so any possible sound has the form

$$s(t) = c_1 \cos(2\pi\nu_1 t) + c_2 \sin(2\pi\nu_1 t) + c_3 \cos(2\pi\nu_2 t) + c_4 \sin(2\pi\nu_2 t),$$

and it can be represented by the coordinate vector

$$[s(t)] = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}.$$

Mathematically, mixing frequencies is simply a matter of forming linear combinations of column vectors with four entries.