A Framework for Interpolation and Approximation on Riemannian Manifolds

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Abstract

In this paper we provide a framework for studying the approximation order resulting from using strictly positive definite kernels to do generalized Hermite interpolation and approximation on a compact Riemannian manifold. We apply this framework to obtain explicit estimates in cases of the circle and 2-sphere. In addition, we provide a technique for constructing strictly positive definite spherical functions out of radial basis functions, and we use it to make a spherical function that is locally supported.

1 Introduction

1.1 Overview

The object of this paper is to provide a brief overview of our recent investigations concerning variational principles and Sobolev-type estimates for the approximation order resulting from using strictly positive definite kernels to do generalized Hermite interpolation on a closed (i.e., no boundary), compact, connected, orientable, m-dimensional C^∞ Riemannian manifold M^m. A complete discussion of the results stemming from these investigations may be found in [1]. The manifolds of particular interest in this article will be the

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circle and the 2-sphere, although our techniques apply to certain manifolds which arise in group theory as well. We point out that in the case of the 2-sphere some work in this direction has already been done [3, 6, 11, 14, 15]. The rates of approximation discussed here will be analyzed in terms of Sobolev norms. We point out that the constants appearing in the approximation order inequalities are explicit. (See §5.1 and §5.2). Finally, we mention that in §3.3 we give some new results that allow one to make make use of radial basis functions (RBFs) on R^{m+1} to obtain strictly positive definite functions on S^{m-1}. We use this to take a compactly supported RBF in R^5 [16] and produce a locally supported, strictly postive definite function on S^m, $m \leq 3$..

1.2 Sobolev Spaces

In this section we briefly describe the notion of a Sobolev space on a C^∞ manifold; for proofs and more references, see [1, 8].

The Riemannian metric g^{ij} on M^m induces the standard measure, $d\mu$, on M^m, and also gives rise to the Laplace-Beltrami operator, Δ, which is elliptic and self-adjoint relative to $L^2(M^m, g)$. The spectrum of $-\Delta$ is discrete, starts at 0, and has $+\infty$ as its only accumulation point; each eigenvalue in this spectrum has finite multiplicity. The eigenfunctions corresponding to these eigenvalues are C^∞-functions. We will label the eigenvalues the with index j that comes from listing the eigenvalues in increasing order $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \ldots$, with appropriate repetitions for degenerate cases. We will let F_j be the eigenfunction corresponding to the eigenvalue λ_j, and we will choose the F_j's so that they form an orthonormal basis for $L^2(M^m, g)$.

The notion of a distribution may be defined on a manifold; as usual it is a linear functional defined on $C^\infty(M^m)$. We shall denote the set of all distributions on the manifold M^m by $\mathcal{D}'(M^m)$. The action of distributions on test functions will be given by

$$(u, \phi) = \int_{M^m} u(p)\phi(p)d\mu(p). \tag{1.1}$$

Since the eigenfunctions F_j are themselves C^∞ functions, we may compute the coefficients in a formal eigenfunction expansion for u:

$$\widehat{u}(j) := (u, \bar{F}_j)\,.$$

The eigenfunction expansion for u can be used to express (1.1) in the form [8, pp. 178-9]

$$(u, \bar{\phi}) = \sum_{j=0}^{\infty} \widehat{u}(j)\overline{\widehat{\phi}(j)}. \tag{1.2}$$

The Sobolev space associated with an arbitrary real number s is

$$\begin{aligned} H_s(M^m) = &\quad \{u \in \mathcal{D}'(M^m) \, : \, \textstyle\sum_{j=1}^{\infty} \lambda_j^s |\widehat{u}(j)|^2 < \infty\}, \\ \text{with norm } \|u\|_s = &\quad \left(\textstyle\sum_{j=0}^{\infty} (1+\lambda_j)^s |\widehat{u}(j)|^2\right)^{1/2}. \end{aligned}$$

The Sobolev embedding theorem applies here, and so $C^k(M^m) \supset H_s(M^m)$ whenever $s > k + \frac{m}{2}$. In addition, the standard duality $H_{-s} = H_s^*$ holds.

2 Positive Definite Kernels

A continuous, complex-valued kernel $\kappa(\cdot,\cdot)$ is termed *positive definite on* M^m if $\bar{\kappa}(q,p) = \kappa(p,q)$ and if for every finite set of points $\mathcal{C} = \{p_1, \ldots, p_n\}$ in M^m, the self-adjoint, $N \times N$ matrix with entries $\kappa(p_j, p_k)$ is positive semi-definite. (See [13, 8, 1].)

One can view this positivity in terms of distributions. If δ_p is the Dirac delta function located at p and if $u = \sum_j c_j \delta_{p_j}$, then the matrix $[\kappa(p_j, p_k)]$ being positive semi-definite is equaivalent to the quadratic form $(\bar{u} \otimes u, \kappa) \geq 0$ for every choice of c_j's. This result extends to a somewhat more general situation. By [1, Proposition 1.3], if u and v are distributions in $H_{-s}(M^m)$, then the tensor product $u \otimes v$ is in $H_{-2s}(M^m \times M^m)$. It follows that when κ is in $H_{2s}(M^m \times M^m)$, $(u \otimes v, \kappa)$ is well defined, and by [1, Theorem 2.1], $(\bar{u} \otimes u, \kappa) \geq 0$. Thus κ is positive definite on $H_{-s}(M^m)$. This "extension theorem" has a corollary that is useful for constructing positive definite kernels on manifolds embedded in R^n (or any other manifold, for that matter).

Corollary 2.1 *If $\kappa(q,p)$ is a continuous positive definite kernel on R^n, and if M^m is embedded in R^n, then the restriction of κ to M^m is positive definite.*

Proof: The matrix $[\kappa(p_j, p_k)]$ is positive semi-definite for any distinct set of points in R^n. It will therefore be positive definite when the points are restricted to M^m. ∎

The mapping properties associated with a kernel $\kappa \in H_{2s} \cap C^0(M^m \times M^m)$ are important to us. Define the linear transformation $\Psi_\kappa : C^0(M^m) \to C^0(M^m)$ via the expression

$$\Psi_\kappa[f](p) = \int_{M^m} \kappa(p,q) f(q) d\mu(q).$$

In [1, Proposition 1.3], we show that Ψ_κ is a bounded linear map from $H_{-s}(M^m)$ into $H_s(M^m)$. To emphasize the role the kernel plays here, we will write

$$\kappa \star f(p) := \psi_\kappa[f](p).$$

3 Strictly Positive Definite Kernels

3.1 Generalized Hermite Interpolation

The interpolation problem that we wish to address is the following. Think of a function f that is smooth, say in H_s, and that we wish to reconstruct from data whose measurement is mathematically modeled as the application of a linear functional to the unknown function f. For example, measuring the value of f at a point p can be thought of as obtaining (δ_p, f). More generally, take $\mathcal{U} := \{u_j\}_{j=1}^N \subset H_{-s}$ be a linearly independent (i.e., non-redundant) set of distributions on M^m, and take the measurements to be

$$\int_{M^m} \bar{u}_j(p) f(p) d\mu(p) = d_j, \quad \text{for } j = 1, \dots, N, \tag{3.1}$$

where the d_j's are complex numbers. (We remark that the complex conjugate appears above to simplify formulas used later on.)

The choice of interpolant requires discussion. In the case of scattered-data interpolation on R^m with an order 0 radial basis function (RBF) F (see [10] for a review), the interpolant has the form

$$\begin{aligned} \tilde{f}(x) &= \textstyle\sum_{j=1}^N c_j F(\|x-\xi_j\|) \\ &= \textstyle\sum_{j=1}^N c_j \left(\delta_{\xi_j}(y), F(\|x-y\|)\right) \\ &= \Big(\underbrace{\sum_{j=1}^N c_j \delta_{\xi_j}(y)}_{u(y)}, F(\|x-y\|)\Big) \\ &= (u(y), F(\|x-y\|)) \\ &= F(\|x-\cdot\|) \star u \,. \end{aligned}$$

This suggests that we look for an interpolant of the form

$$\tilde{f} := \kappa \star u, \text{ where } u \in \mathcal{U}. \tag{3.2}$$

3.2 Strict Positivity and an Inner Product for H_{-s}

Now that we have settled on the form of the interpolant, we need to find the kernels for which one always has that there is a $u \in \mathcal{U}$ such that (3.1) is satisfied by $\tilde{f}$ in (3.2). By standard arguments, this is equivalent to the interpolation matrix A with enries,

$$\begin{aligned} A_{j,k} &= (\bar{u}_j, \kappa \star u_k) \\ &= (\bar{u}_j \otimes u_k, \kappa), \end{aligned}$$

being invertible. The quadratic form associated with A is

$$c^* A c = (\bar{u} \otimes u, \kappa), \text{ where } u = \sum_{j=1}^N c_j u_j \tag{3.3}$$

If κ is positive definite, then, by our earlier remarks in §2, $c^* A\, c \geq 0$; thus, A is positive semi-definite. This is still not sufficient for A to be invertible under all circumstances. For that, we will require that the kernel κ satisfy

$$(\bar{u} \otimes u, \kappa) > 0\,,\ \forall\ u \neq 0,\ u \in H_{-s}(M^m). \tag{3.4}$$

If (3.4) holds, we will say that κ is *strictly positive definite*. Since a strictly positive definite kernel means a positive definite interpolation matrix for any of the interpolation problems described in §3, all of these problems are well poised.

One can use strictly positive definite kernels that are continuous and in $H_{2s}(M^m \otimes M^m)$ to define an inner product on H_{-s}; simply let

$$\llbracket u, v \rrbracket := (\bar{v} \otimes u, \kappa) \tag{3.5}$$

$$\llbracket u \rrbracket := (\bar{u} \otimes u, \kappa)^{1/2}. \tag{3.6}$$

We note that although one could complete H_{-s} to a Hilbert space using the inner product (3.5), it is not necessary to do so.

Strictly positive definite kernels have the property that all of the matrices $[\kappa(p_j, p_k)]$ are positive definite for an arbitrary finite set of distinct points $\{p_1, \ldots, p_N\}$. It is tempting to think that the converse is true. Sad to say, there is a counterexample; see the paper by Ron and Sun [11].

3.3 Examples

Our basic source of examples is the following theorem.

Theorem 3.1 *Let $\kappa(p,q)$ be a continuous kernel in $H_{2s}(M^m \times M^m)$ having the eigenfunction expansion*

$$\kappa(p,q) := \sum_{\alpha \in \mathcal{A}} a(\alpha) F_\alpha(p) \bar{F}_\alpha(q), \tag{3.7}$$

where $\mathcal{A}$ is some countable index set, and the F_α's are the normalized eigenfunctions of the Laplace-Beltrami operator on M^m. κ is strictly positive definite if and only if $a(\alpha) > 0$ for all $\alpha \in \mathcal{A}$.

Corollary 3.2 *If the strictly positive definite kernels $\kappa_1 \ldots, \kappa_n$ all have the form (3.7) on $M^{m_1}, \ldots, M^{m_n}$, then $\kappa = \kappa_1 \cdots \kappa_n$ is strictly positive definite on the product manifold $M^{m_1} \times \cdots \times M^{m_n}$.*

Here are a few explicit examples of such kernels for various manifolds.

1. (M^m) $\kappa_t(p,q) = \sum_{j=1}^{\infty} \exp(-t\lambda_j) F_j(p) \bar{F}_j(q)$, the heat kernel on M^m.

2. (S^m) $\kappa(p,q) = \sum_{\ell,j} a(\ell,j) Y_{\ell,j} \bar{Y}_{\ell,j}$, where the $Y_{\ell,j}$ are the spherical harmonics on S^m.

3. (S^m) $\kappa(p,q) = \sum_{j=0}^{\infty} b_j \cos^j(\theta(p,q))$, $\quad b_j > 0$, where $\theta(p,q)$ denotes the length of the shorter of the two arcs on the great circle joining p and q. The dimension m can be arbitrary.

4. (S^m) $\kappa(p,q) = \exp(s \cos(\theta(p,q))$, $\quad s > 0$ (m arbitrary.)

5. (S^3) $\kappa(p,q) = \left(1 - 2\sin(\frac{\theta(p,q)}{2})\right)_+^5 \left(10\sin(\frac{\theta(p,q)}{2}) + 1\right)$ (See Remark 3.4).

6. (T^m) $\kappa(\phi,\phi') = K(\phi - \phi') = \sum_{\alpha\in Z^m} a(\alpha)e^{i\alpha\cdot(\phi-\phi')}$

7. (T^m) $K(\phi) = \exp(\sum_{j=1}^m s(j)\cos(\phi_j))$, $\quad s_j > 0$

8. (T^m) $K(\phi) = \prod_{1\le j\le m} g(\phi_j)$, where g is an even, 2π-periodic function. On $[0,\pi]$, g is defined this way:

$$g(\phi) = \begin{cases} a^{-5}(a-\phi)^3(a^2+3a\phi+\phi^2) & \text{if } 0 \le x \le a \\ 0 & \text{if } a < \phi \le \pi. \end{cases}$$

Let $f(x) = F(\|x\|)$ be a smooth, order-0 RBF defined on R^{m+1}. If q and p are points on the unit S^m embedded in R^{m+1}, we recall that

$$\|p-q\|^2 = 2 - 2p\cdot q = 2 - 2\cos(\theta),$$

where $p\cdot q$ is the Euclidean dot-product and θ is the angle between p and q. Define the function

$$G(\cos(\theta)) := F\left(\sqrt{2-2\cos(\theta)}\right). \tag{3.8}$$

By virtue of Corollary 2.1, $G(p\cdot q) = F(\|p-q\|)$ is a positive definite kernel on S^m. Moreover, since G is a function of $\cos(\theta)$ alone, it is also a positive definite spherical *function* [12], and so admits an expansion in Legendre polynomials [7, p. 21] appropriate to S^m,

$$G(\cos(\theta)) = \sum_{\ell=0}^{\infty} g_m(\ell)P_\ell(m+1;\cos(\theta)) \tag{3.9}$$

with $g_m(\ell) \ge 0$. By [9, Proposition 2.1], if $g_m(\ell) > 0$ for infinitely many even ℓ and infinitely many odd ℓ, then it is strictly positive definite on S^{m-1}. Alternatively, if we let $t = \cos(\theta)$, then neither the even part nor the odd part of $G(t)$ is a polynomial. We summarize these remarks below.

Corollary 3.3. *If F is an order-0 RBF on R^{m+1}, then G is a positive definite function on S^m. Moreover, if neither of the two functions,*

$$G(\cos(\theta)) \pm G(-\cos(\theta))$$

is a polynomial in $\cos(\theta)$, then G is strictly positive definite on S^{m-1}.

This corollary allows us to draw upon some of the new, compactly supported RBFs that Wendland [16] has constructed. Although these RBFs have the slight disadvantage of being dimension dependent, their having compact support more than makes up for it. One of these, the RBF below, is positive definite on R^5 and is C^2.

$$\phi_{5,1}(r) := (1-r)_+^5(5r+1)\,. \tag{3.10}$$

Using the prescription above, we first let $r = \sqrt{2-2\cos(\theta)}$, and then set

$$\begin{aligned}\gamma(\cos(\theta)) :=&\ \phi_{5,1}\left(\sqrt{2-2\cos(\theta)}\right) && (3.11)\\ =&\ \left(1-\sqrt{2-2\cos(\theta)}\right)_+^5)(1+5\sqrt{2-2\cos(\theta)}) && (3.12)\\ =&\ \left(1-2\sin(\tfrac{\theta}{2})\right)_+^5\left(10\sin(\tfrac{\theta}{2})+1\right)\,. && (3.13)\end{aligned}$$

Remark 3.4. *The function $\gamma(\cos(\theta))$ defined above is strictly positive definite on S^m for $m = 1$, 2, and 3.*

Proof: Observe that $r = \sqrt{2-2\cos(\theta)} = 2\sin\left(\frac{\theta}{2}\right)$ is larger than 1 for $\pi/3 \le \theta \le \pi$. Hence, $\gamma(\cos(\theta)) = 0$ for such θ. On the other hand, replacing $\cos(\theta)$ by $-\cos(\theta)$ gives us $r = \sqrt{2+2\cos(\theta)} = 2\cos\left(\frac{\theta}{2}\right)$, which exceeds 1 on the interval $0 \le \theta \le 2\pi/3$. Hence, $\gamma(-\cos(\theta)) = 0$ on the interval $0 \le \theta \le 2\pi/3$. It follows that the even and odd parts of $\gamma(\cos(\theta))$ vanish identically for $\pi/3 \le \theta \le 2\pi/3$, and so neither can be polynomials in $\cos(\theta)$. Of course, γ is not identically 0; thus, γ is strictly positive definite on S^m with $m \le 5-2 = 3$. ∎

4 A Variational Framework for Interpolation

4.1 The Variational Principle and Basic Bounds

We will assume at the outset that κ is strictly positive definite, continuous, and in $H_{2s}(M^m \times M^m)$. The generalized Hermite interpolation problem described in §3.1, equation (3.1), can be recast in terms of the inner product defined by (3.5). We assume that the data is generated by applying distributions from the set $\{u_j\}_{j=1}^N \subset H_{-s}$ to an unknown $H_s(M^m)$ function of the form $f = \kappa \star v$, with v is in $H_{-s}(M^m)$. That is, we assume the data set has the form

$$(\bar{u}_j, f) = [\![v, u_j]\!] = d_j, \quad \text{for } j = 1, \ldots, N. \tag{4.1}$$

Let $\mathcal{U} = \text{Span}\left(\{u_j\}_{j=1}^N\right)$. We seek to interpolate the data and approximate the function $f = \kappa\star v$ with a function of the form $\tilde{f} = \kappa\star u$ for some $u \in \mathcal{U}$. As we mentioned earlier, the interpolation matrix for this problem is invertible, and so there exists a unique $u \in \mathcal{U}$ that solves it. The distribution u satisfies the following minimization principle.

Theorem 4.1 [1, Theorem 3.1]: *Let $u \in H_{-s}(M^m)$ be the unique distribution in $\mathcal{U}$ for which $\kappa \star u$ satisfies*

$$[\![u, u_j]\!] = d_j, \quad \text{for } j = 1, \dots, N.$$

If $v \in H_{-s}(M^m)$ solves (4.1) then $v - u$ is orthogonal to $\mathcal{U}$ with respect to the inner product (3.5). In addition,

$$[\![v]\!]^2 = [\![v - u]\!]^2 + [\![u]\!]^2.$$

Finally, if $v \neq u$,

$$[\![u]\!] < [\![v]\!].$$

We remark that this is similar to the variational principle derived by Madych and Nelson [4, 5] in the radial basis function (RBF) case. The last inequality in the theorem amounts to saying that among all $v \in H_{-s}(M^m)$ for which $\kappa \star v$ interpolates the data, the distribution u minimizes the norm (3.6). Since both f and $\tilde{f}$ satisfy (4.1), one has that

$$(\bar{u}_j, f - \tilde{f}) = [\![v - u, u_j]\!] = 0 \text{ for } j = 1, \dots, N,$$

and so $v - u$ is in $\mathcal{U}^\perp$. Clearly, u is the orthogonal projection of v onto $\mathcal{U}$. Next, let w be an arbitrary distribution in $H_{-s}(M^m)$. We wish to bound the quantity $(\bar{w}, f - \tilde{f})$; such a bound is useful in getting rates of approximation. Rewriting this quantity as

$$(\bar{w}, f - \tilde{f}) = [\![v - u, w]\!],$$

and using the fact that $v - u \in \mathcal{U}^\perp$ together with standard Hilbert space methods, we obtain [1, Proposition 3.2]

$$|(\bar{w}, f - \tilde{f})| \leq \operatorname{dist}(v, \mathcal{U}) \operatorname{dist}(w, \mathcal{U}) . \tag{4.2}$$

The utility in (4.2) is that we can estimate the effect of w applied to $f - \tilde{f}$ independently of the original f. Also, the distributions w and v appear symmetrically on the right-hand side above, in factors that involve only distances to $\mathcal{U}$. Thus our strategy for obtaining estimates on $|(\bar{w}, f - \tilde{f})|$ is to get upper bounds on the distance from a distribution to $\mathcal{U}$, given that the norm we employ is (3.6).

4.2 A Framework for Distance Estimates

We now restrict our attention to kernels of the form (3.7); for the sake of simplicity, we will assume the index set $\mathcal{A}$ in (3.7) is the set of nonnegative integers. For such kernels, we can use the framework below to help estimate the distance from a distribution in H_{-s} to $\mathcal{U}$.

Proposition 4.2 [1, Proposition 3.6]: *Let M be a positive integer, let $s > 0$, and let κ have the form (3.7) with $a(k) > 0$ for all $k \geq 0$. If there are coefficients $c_1, \ldots, c_N$ such that, for $k = 0 \ldots M$, $(w - \sum_{j=1}^{N} c_j u_j, \bar{F}_k) = 0$, and if there is a sequence $b(k) > 0$, $k = M+1, M+2, \ldots$, for which $|(w - \sum_{j=1}^{N} c_j u_j, \bar{F}_k)|^2 \leq b(k)$ when $k \geq M+1$, then*

$$\operatorname{dist}(w, \mathcal{U}) \leq [\![w - \sum_{j=1}^{N} c_j u_j]\!] \leq \left(\sum_{k=M+1}^{\infty} a(k) b(k) \right)^{1/2}. \tag{4.3}$$

In addition to this proposition, we will use another result that connects the norm (3.6) with more familiar Sobolev norms. This result will apply to kernels of the form (3.7) for which the $a(k)$'s do not decay too rapidly. We remark that the two results are independent of one another.

Proposition 4.3 [1, Proposition 3.5]: *Let s, κ, and the $a(k)$'s be as in Proposition 4.2. If there is a $t > s$ for which*

$$a(k) \geq c^{-1} (1 + \lambda_k)^{-(s+t)/2} \tag{4.4}$$

for all k and some constant c, then every $f \in H_t(M^m)$ can be written as $f = \kappa \star v$ for some v in $H_{-s}(M^m)$. Moreover,

$$\|v\|_{-s} \leq c \|f\|_t \quad \text{and} \quad [\![v]\!] \leq \|\kappa\|_{2s}^{1/2} c \|f\|_t . \tag{4.5}$$

We now turn to examples in which the framework established here is applied.

5 Applications

5.1 Tori

We will explain how our framework is used in a simple case: interpolating data at equally spaced points on the unit circle, T^1. In that case, we have that the distributions are $u_j = \delta_{2\pi j/N}$, where $j = 0, \ldots, N-1$. The eigenfunctions are of course

$$F_k(\phi) = \frac{e^{ik\theta}}{\sqrt{2\pi}}.$$

For our kernel we will use

$$\kappa(\phi, \phi') := \sum_{k=-\infty}^{\infty} a(k) e^{ik(\phi - \phi')},$$

which is of the form (3.7). We assume that we are given $w \in H_{-s}(T^1)$, and that the $a(k)$'s decay sufficiently fast for P to be in H_{2s}. We will take $M = N - 1$.

To apply Proposition 4.2 to his case, we must find coefficients c_j, $j = 0, \ldots, N-1$, such that $(w - \sum_{j=0}^{N-1} c_j u_j, e^{-ik\phi}/\sqrt{2\pi}) = 0$. In terms of the (usual) Fourier coefficients for w, we have

$$\widehat{w}(k) = \underbrace{\sum_{j=0}^{N-1} c_j e^{-2\pi i k j/N}}_{\widehat{c}_k},$$

where k runs from $-[N/2]+1$ to $[(N-1)/2]$ and $\widehat{c}_k$ is the discrete Fourier transform (DFT) of the c_j's. Inverting the DFT yields the c_j's. It is easy to obtain $b(k)$'s so that $|(w - \sum_{j=0}^{N-1} c_j u_j, e^{-ik\phi}/\sqrt{2\pi})| \leq b(k)$ for other values of k. The estimate on $\mathrm{dist}(w, \mathcal{U})$ from Proposition 4.2 is

$$\mathrm{dist}(w, \mathcal{U}) \leq 2\|w\|_{-s} \left(\sum_{k \notin \mathcal{I}_N} (1+k^2)^s a_k \right)^{1/2},$$

where $\mathcal{I}_N := [-[N/2], [(N-1)/2]\,] \cap Z$. Using the inequality above in conjuction with (4.2) yields

$$|(\bar{w}, f - \tilde{f})| \leq 4\|v\|_{-s}\|w\|_{-s} \left(\sum_{k \notin \mathcal{I}_N} (1+k^2)^s a_k \right).$$

If the coefficients $a(k)$ decay fast enough for the series to converge, but obey the condition in Proposition 4.3, then $\|v\|_{-s} \leq c\|f\|_t$ and hence

$$|(\bar{w}, f - \tilde{f})| \leq 4c\|f\|_t\|w\|_{-s} \left(\sum_{k \notin \mathcal{I}_N} (1+k^2)^s a_k \right).$$

This final bound is useful if one has *a priori* information about the "energy" of the function f generating the data. These results extend to u_j's that are more general than point evaluations, include data that are only quasi-uniformly distributed, and carry over to some extent to T^m. See [1, §5].

5.2 The 2-Sphere

In the interpolation problem that we wish to deal with here, we will use the physicist's convention of taking ϕ to be the azimuthal angle and θ the angle measured off the z-axis. The distributions will be point evaluations. Let Λ be a fixed positive integer, and let $p_{j,k} \in S^2$ have coordinates (θ_j, ϕ_k), where $\theta_j = \frac{\pi j}{2\Lambda}$, $\phi_k = \frac{\pi k}{\Lambda}$, and $j, k = 0, \ldots, 2\Lambda - 1$. We then take $u_{j,k} = \delta_{p_{j,k}}$.

The eigenfunctions for the Laplace-Beltrami operator on the 2-sphere are the spherical harmonics [7], $Y_{\ell,m}$, where $\ell = 0, 1, \ldots$ and $m = -\ell, \ldots, \ell$. We will use

$$\kappa(p,q) := \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a(\ell,m) Y_{\ell,m}(p) \overline{Y_{\ell,m}(q)}.$$

Again these are of the form (3.7). As before, we assume that the $a(\ell,m)$'s decay quickly enough for κ to be in $H_{2s}(S^2 \times S^2)$.

If Λ is a power of 2, one can use Proposition 4.2 and a spherical version of the FFT [2] to get bounds on $\mathrm{dist}(w,\mathcal{U})$ [1, Theorem 6.6] similar to those given above in §5.1. We can use these bounds in (4.2) to obtain [1, Corollary 6.7]

$$|(\bar{w}, f - \tilde{f})| \leq \Sigma(\Lambda)^2 \|v\|_{-s} \|w\|_{-s} \left(\sum_{\ell=\Lambda}^{\infty} \sum_{m=-\ell}^{\ell} (1 + \ell(\ell+1))^s a(\ell, m) \right)$$

where $\Sigma(\Lambda) := 1 + \frac{1}{\sqrt{\pi}} \Lambda^{\frac{3}{2}} \log_2(16\Lambda)$. Of course, if the $a(\ell,m)$'s satisfy (4.4), then by Proposition 4.3 we may replace $\|v\|_{-s}$ by $c\|f\|_s$. For further details, proofs, and precise statements of results, see [1, §6].

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