The DFT Approximation to the Fourier Transform

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12 November 1998

1 Introduction

In these notes we simply wish to show how one may compute a numerical approximation to the Fourier transform of a function f from samples taken at N grid points spaced T apart, starting at a time $t = a$ and ending at $t = b - T$, where $b = a + NT$. (Including $t = b$ gives $N + 1$ samples, one more than we have.) We will assume that a function f is 0 outside of the interval $[a, b]$, continuous on $[a, b]$, and that it satisfies $f(b) = f(a)$. In practice, the function will not have been sampled at $t = b$; more often than not, our assumption will result in a Gibbs' phenomenon occurring when one reconstucts the function from the data.

2 Fourier series and $\hat{f}$

The Fourier transform of f is given by

$$\hat{f}(\omega) = \int_a^b f(t)e^{-i\omega t}dt$$

We want to change variables so that the time interval runs from 0 to 2π rather than a to b. Change variables to $\theta = 2\pi\frac{t-a}{b-a}$; $\hat{f}$ is then given by

$$\begin{aligned}\hat{f}(\omega) &= \frac{b-a}{2\pi}\int_0^{2\pi} f(a+(b-a)\theta/(2\pi))e^{-i\omega(a+(b-a)\theta/(2\pi))}d\theta \\ &= \frac{b-a}{2\pi}e^{-i\omega a}\int_0^{2\pi} f(a+(b-a)\theta/(2\pi))e^{-i\frac{(b-a)\omega}{2\pi}\theta}d\theta.\end{aligned}$$

Note that if we define the function g and the frequency ω_k via

$$g(\theta) := f(a+(b-a)\theta/(2\pi)) \quad \text{and} \quad \omega_k = \frac{2\pi}{b-a}k$$

then $\hat{f}(\omega_k)$ has the form

$$\begin{aligned}\hat{f}(\omega_k) &= (b-a)e^{-i\omega_k a}\left\{\frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-ik\theta}d\theta\right\} \\ &= (b-a)e^{-i\omega_k a}\hat{g}_k,\end{aligned}$$

where $\hat{g}_k$ is the kth Fourier series coefficient for g.

3 DFT and $\hat{f}$

The relationship between the DFT and Fourier series coefficients is one that we have dealt with before. First, let

$$y_j := g(2\pi j/N) = f(a + j\underbrace{\frac{b-a}{N}}_{T}),\ j = 0\ldots N-1.$$

Thus the y_j's are the samples of f taken at times $a+Tj$. The DFT approximation to $\hat{g}_k$ is $\hat{y}_k/N$. This implies that the DFT for the y_j's is related to $\hat{f}$ via

$$\hat{f}(\omega_k) \approx \frac{b-a}{N}e^{-i\omega_k a}\hat{y}_k = Te^{-i\omega_k a}\hat{y}_k\,.$$