Properties of the Fourier Transform

Definition: $\hat{f}(\lambda) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx.$

- Notation: $\mathcal{F}[f](\lambda) := \hat{f}(\lambda)$.

Immediate consequences of the definition:

- Linearity: $\mathcal{F}[f+g] = \mathcal{F}[f] + \mathcal{F}[g] \quad \mathcal{F}[cf] = c\mathcal{F}[f]$.
- Continuity of the FT: *If* $\int_{-\infty}^{\infty} |f(x)|dx < \infty$, *then* $\hat{f}$ *is continuous in* λ.
- Important elementary formulas & facts:
  (1) Multiplication by x^n: $\mathcal{F}[x^n f] = i^n \hat{f}^{(n)}$.
  (2) Derivatives: $\mathcal{F}[f^{(n)}] = (i\lambda)^n \hat{f}$.
  (3) Translations: $\mathcal{F}[f(x-a)] = e^{-i\lambda a}\hat{f}(\lambda)$.
  (4) f *is real-valued if and only if* $\overline{\hat{f}(\lambda)} = \hat{f}(-\lambda)$.
  (5) f *is even if and only if* $\hat{f}(\lambda) = \hat{f}(-\lambda)$.
  (6) f *is odd if and only if* $\hat{f}(\lambda) = -\hat{f}(-\lambda)$.

Inversion formula: $\dfrac{f(x^+) + f(x^-)}{2} = \displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{i\lambda x}d\lambda.$

- Notation: $\mathcal{F}^{-1}[g](x) = \displaystyle\int_{-\infty}^{\infty} g(\lambda)e^{i\lambda x}d\lambda$.
- Linearity: $\mathcal{F}^{-1}[f+g] = \mathcal{F}^{-1}[f] + \mathcal{F}^{-1}[g] \quad \mathcal{F}^{-1}[cf] = c\mathcal{F}^{-1}[f]$.
- Important formulas & facts:
  (1) Derivatives: $\mathcal{F}^{-1}[\hat{f}^{(n)}] = (-ix)^n f$
  (2) Multiplication by λ^n: $\mathcal{F}^{-1}[\lambda^n \hat{f}] = (-i)^n f^{(n)}$.
  (3) Multiplication by $e^{-i\lambda a}$: $\mathcal{F}^{-1}[e^{-i\lambda a}\hat{f}](x) = f(x-a)$.
  (4) $\mathcal{F}^{-1}[g](z) = 2\pi\mathcal{F}[g](-z)$

Convolution product: $f * g(x) := \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(u)g(x-u)du.$

Convolution Theorem: $\mathcal{F}[f*g] = \hat{f}\hat{g}$ and $\mathcal{F}^{-1}[\hat{f}\hat{g}] = f * g$

Parseval's Formula: $\displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda)\overline{\hat{g}(\lambda)}d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$