# Test II

**Instructions:** *Show all work in your bluebook. Calculators that do linear algebra or calculus are* not *allowed.*

1. **(10 pts.)** Define these: (a) Linearly independent vectors. (b) Basis.

2. **(20 pts.)** For the matrix $A$ below, find bases for the column space, row space and null space of $A$. What are the rank and nullity of $A$?

$$A = \begin{pmatrix} 1 & -3 & -1 & -3 \\ -1 & 3 & 2 & 4 \\ 2 & -6 & 4 & 0 \end{pmatrix}$$

3. **(15 pts.)** Determine whether each of the sets below is linearly dependent or linearly independent, and whether or not it is a basis for $P_4$.

$$\begin{aligned} S_1 &= \{1 - x, 1 + x^2, 1 - x + x^3, 1 + x^2 - 5x^3, x - x^3\} \\ S_2 &= \{x + 1, 1 - x, x^2 + x^3\} \\ S_3 &= \{1 + x^2, 1 - x^3, x, 1 - x^2\}. \end{aligned}$$

4. **(15 pts.)** $E = \{1 - x, 2x + 1\}$ and $F = \{1 + x, x - 2\}$ are bases for $P_2$. Find the transition matrix $S_{E\to F}$. (Hint: $E$ and $F$ can be expressed in terms of a *third* basis, $G = \{1, x\}$.)

5. Let $L : P_3 \to P_3$ be defined by $L(p) = p'' + (x + 1)p' - p$.

   (a) **(5 pts.)** Show that $L$ is linear.

   (b) **(5 pts.)** Find the matrix of $L$ relative to the basis $E = \{1, x, x^2\}$.

   (c) **(10 pts.)** Find $p \in P_3$ that solves the differential equation $L(p) = 4 + 2x + x^2$.

6. **(15 pts.)** Do <u>one</u> of these.

   (a) **(10 pts.)** Show that if $V = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_\mathrm{n})$ , where $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, then $\mathbf{v} \in V$ can be written in only one way as a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

   (b) Prove that a linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathrm{rank(A)} = \mathrm{rank(A|}\mathbf{b})$.

   (c) Let $L : V \to V$ be linear, and suppose that $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ are bases for $V$. If $A_E$ and $A_F$ are matrices representing $L$ relative $E$ and $F$, show that $A_E$ and $A_F$ are similar.