Test III

Instructions: *Show all work in your bluebook. Calculators that do linear algebra or calculus are* not *allowed.*

1. **(10 pts.)** Find the eigenvalues and eigenvectors of $C = \begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$. Explain why C is *not* diagonalizable.

2. In the following problem, $A = \begin{pmatrix} -4 & 3 \\ 3 & -12 \end{pmatrix}$.

   (a) **(15 pts.)** Find the eigenvalues and eigenvectors of A and a matrix X such that $X^{-1}AX$ is diagonal.

   (b) **(10 pts.)** Let $\mathbf{x}(t) = [x_1(t), x_2(t)]^T$, where x_1 and x_2 are the equilibribrium displacements for the masses in the spring system below. Newton's law and Hooke's law yield the equation $m\ddot{\mathbf{x}} = kA\mathbf{x}$. Use the answer to part (a) to find the normal modes for the system.

3. Consider the vector field $\mathbf{F}(\mathbf{x}) = 2xy\mathbf{i} + (x^2 + 2yz)\mathbf{j} + y^2\mathbf{k}$.

   (a) **(15 pts.)** Find the derivative matrix $D\mathbf{F}$, the divergence $\nabla \cdot \mathbf{F}$, and the curl $\nabla \times \mathbf{F}$.

   (b) **(5 pts.)** Is $\mathbf{F}$ conservative – i.e., is $\mathbf{F} = \nabla f$ for a scalar valued function?

4. **(10 pts.)** Let $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R}^3 \to \mathbb{R}$, and $h = g \circ f : \mathbb{R}^2 \to \mathbb{R}$, where

$$f(x, y) = (xy, 3x - 2y, x^2y)^T \text{ and } g(u, v, w) = uw + v^2 + w^2$$

   Use the chain rule to find the direction in which h is *increasing* most rapidly at $x = y = 1$.

5. Let $(x, y) = T(u, v) = (u + 2uv, uv)$, and let D^* be the u-v rectangle $1 \le u \le 2$, $0 \le v \le 1$.

   (a) **(5 pts.)** Make a rough sketch of the image region $D = T(D^*)$.

   (b) **(10 pts.)** Use Jacobi's Theorem to change variables and compute the double integral $\int_D \cos\left(\frac{\pi y}{x - 2y}\right) dxdy$.

6. **(20 pts.)** Let $\mathbf{G}(\mathbf{x}) = 2y\mathbf{i} - 3x\mathbf{j}$ and let C be the circle $x^2 + y^2 = 1$. Use the divergence form of Green's Theorem to show that $\oint_C \mathbf{G} \cdot \mathbf{n} ds = 0$, then verify this by direct computation of the line integral.