Test II

Instructions: *Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed.*

1. **(10 pts.)** Determine whether or not $S = \{f \in C[2,5] \,|\, f(2) = f(5)\}$ is a subspace of $C[2,5]$.

2. **(5 pts.)** For the subsets of $\mathcal{P}_3$ below, which *cannot* be linearly independent? Which *cannot* span? Briefly explain how you are getting your answers.

   (a) $\{1-2x, x^2, x+x^3\}$
   (b) $\{x^2, x^3, 1-3x+10x^2-5x^3\}$
   (c) $\{1, x^2-2, x^2-6\}$
   (d) $\{x-3x^2, x^3, x^2-3x+2, x-1, 5x^2-7\}$
   (e) $\{1, x, x^3, x-x^2, 3x-1, 2x^3\}$.

3. **(10 pts.)** Let $B = \{1, e^x, e^{2x}\}$. Show that B is LI and find the coordinates of $f(x) = (1-3e^x)^2$.

4. **(15 pts.)** Find bases for the column space, null space, and row space of C, and state the rank and nullity of C. What should these sum to? Do they?

$$C = \begin{pmatrix} 1 & -2 & -1 & -3 \\ -1 & 2 & 2 & 4 \\ 2 & -4 & -3 & -7 \end{pmatrix}$$

5. Let $L : \mathcal{P}_2 \to \mathcal{P}_2$ be defined by $L[p] = (x^2-1)p'' + (x+3)p' - 4p$.

   (a) **(5 pts.)** Show that L is linear.
   (b) **(5 pts.)** Find the matrix A of L relative to the basis $B = \{1, x, x^2\}$.
   (c) **(5 pts.)** Find a basis for the null space of L. What is the rank of L?

6. **(15 pts.)** Consider the matrix A below. Find the eigenvalues and eigenvectors of A. Also, find a diagonal matrix Λ and an invertible matrix S for which $A = S\Lambda S^{-1}$.

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

7. **(15 pts.)** Consider the inner product $\langle f, g \rangle = \int_0^\pi f(x)g(x)dx$ on continuous functions $C[0, \pi]$. Use the Gram-Schmidt process to turn $\{1, \cos(x), \cos^2(x)\}$ into an orthogonal set relative to this inner product. (Hint: $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$.)

8. Consider the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$ on continuous functions $C[-1, 1]$.

   (a) **(5 pts.)** Find the Gram matrix A for $\{1, x, x^2\}$.

   (b) **(10 pts.)** Use A and the given inner product to find the best quadratic fit to $f(x) = |x|$.