Quiz 1 – Key

Instructions: Show all work in the space provided. No notes, calculators, cell phones, etc. are allowed.

1. Define the terms below.

   (a) **(5 pts.)** function f from a set X to a set Y – p. 2.

   (b) **(5 pts.)** well-ordering principle – p. 13.

2. **(15 pts.)** Prove that if $|x| \leq 1$, then $|x^2 - x - 2| \leq 3|x + 1|$

   Solution. Note that $|x^2 - x - 2| = |(x - 2)(x + 1)| = |x - 2|\,|x + 1|$. By the triangle inequalty and $|x| \leq 1$, we have $|x - 2| \leq |x| + 2 \leq 3$. Hence, $|x^2 - x - 2| \leq 3|x - 1|$.

3. **(10 pts.)** Use the binomial theorem to show that if a and b are nonnegative real numbers, then $(a + b)^n \geq a^n + na^{n-1}b$.

   Solution. The binomial theorem gives us this chain:

$$\begin{aligned}(a+b)^n &= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \\ &= a^n + na^{n-1}b + \text{ nonneg. terms} \\ &\geq a^n + na^{n-1}b.\end{aligned}$$

4. **(15 pts.)** **(Approximation Property for Suprema)**. Prove this: If $E \subset \mathbb{R}$ has a supremum s, then for every $\varepsilon > 0$ there is an $a \in E$ such that $s - \varepsilon < a \leq s$.

   Proof. Suppose not. Then for some $\epsilon_0 > 0$ the interval $(s - \epsilon_0, s]$ contains no points from E. Since s is the supremum for E, there are no points of E in (s, ∞), either. It follows that all $a \in E$ are in $(-\infty, s - \epsilon_0]$. Hence, $s - \epsilon_0$ is an upper bound for E. However, $s - \epsilon_0 < s$. This is a contradiction, since every upper bound for E is greater than or equal to s, the supremum.