Final Examination

Instructions: *Show all work in your bluebook. You may use a calculator for numerical computations. You may* not *use a graphing calculator or a calculator that can do symbolics.*

1. **(15 pts.)** Let A be the circulant matrix below, and let $a = (-2, 1, 0, 1)$. Verify that if $y = Ax$, then $y = a * x$. Use the DFT and the (circular) convolution theorem to find the eigenvalues of A. (Hint: $\overline{w} = -i$ in this case.)

$$A = \begin{pmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 1 & -2 \end{pmatrix}$$

2. **(20 pts.)** For $j \in \mathbb{Z}$, let V_j be the subspace of $f \in L^2$ such that the support of $\hat{f}$ being in $[-2^j\pi, 2^j\pi]$, and let $\phi(x) = \mathrm{sinc}(x)$; note that $\phi \in V_0$. List all of the properties of a *multiresolution analysis* (MRA). Pick any three; show that the V_j's and ϕ satisfy them.

3. **(15 pts.)** For any MRA, the reconstruction formula is given by $a^j_k = \sum_{\ell\in\mathbb{Z}} p_{k-2\ell}\, a^{j-1}_\ell + \sum_{\ell\in\mathbb{Z}} (-1)^k \overline{p_{1-k+2\ell}}\, b^{j-1}_\ell$. Put this formula in terms of the discrete filters shown in Figure 1. State what $\widetilde{H}$, $\widetilde{L}$, and $2\uparrow$ are. For the Daubechies wavelet, are these filters IIR or FIR?

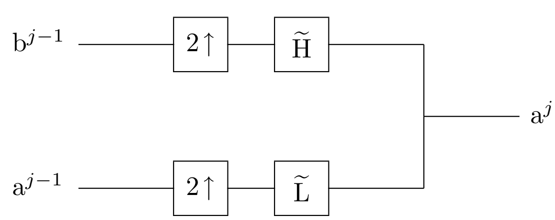

Figure 1: Wavelet reconstruction diagram

4. **(20 pts.)** Let $\phi(x)$ be the Haar scaling function. Use the Haar MRA to decompose to level 0 the function $f \in V_2$, where

$$f(x) = 2\phi(4x + 1) - \phi(4x) + 3\phi(4x - 1) + 5\phi(4x - 2) - \phi(4x - 3).$$

5. **(15 pts.)** The scaling relation for an MRA is $\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x - k)$. Show that this becomes, in the Fourier transform picture, $\hat{\phi}(\xi) = P(e^{-i\xi/2})\hat{\phi}(\xi/2)$, where $P(z) = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_k z^k$. Briefly describe how $\hat{\phi}(\xi)$ is constructed from the function $P(\xi)$. What conditions should $P(z)$ satisfy for this construction to yield a scaling function?

6. **(15 pts.)** For the Daubechies $N = 2$ MRA, the function $P(z)$ is a polynomial,

$$P(z) = (1+z)^2 \left(\frac{1+\sqrt{3}}{8} + \frac{1-\sqrt{3}}{8} z \right).$$

Use this and the formula $\hat{\psi}(\xi) = -e^{-i\xi/2}\overline{P(-e^{-i\xi/2})}\hat{\phi}(\xi/2)$ to show that the Daubechies wavelet has two vanishing moments. Briefly discuss the significance of this for singularity detection.

Properties of the Fourier Transform

1. $\hat{f}(\xi) = \mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx.$
2. $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi.$
3. $\mathcal{F}[x^n f(x)](\xi) = i^n \hat{f}^{(n)}(\xi).$
4. $\mathcal{F}[f^{(n)}(x)](\xi) = (i\xi)^n \hat{f}(\xi).$
5. $\mathcal{F}[f(x-a)](\xi) = e^{-i\xi a} \hat{f}(\xi).$
6. $\mathcal{F}[f(bx)](\xi) = \frac{1}{b} \hat{f}(\frac{\xi}{b}).$
7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\xi) \hat{g}(\xi)$
8. $\mathrm{sinc}(x) := \frac{\sin(\pi x)}{\pi x} = \mathcal{F}^{-1}[\chi_\pi]$, where $\chi_\pi(\xi) = \begin{cases} 1/\sqrt{2\pi}, & -\pi \le \xi \le \pi \\ 0, & |\xi| > \pi \end{cases}.$