Test I

Instructions: *Show all work in your bluebook. You may use a calculator for numerical computations. You may* not *use a graphing calcular or a calculator that can do symbolics.*

1. **(15 pts.)** For $n > 0$, let $f_n(t) = \begin{cases} 1, & 0 \le t \le 1/n\,, \\ 0, & \text{otherwise.} \end{cases}$ Show that $f_n \to 0$ in $L^2[0,1]$. Show that f_n does *not* converge to zero uniformly on $[0,1]$.

2. **(20 pts.)** Use least squares to fit a straight line to the datra below.

x	1	2	4	5
y	7.0	4.2	−2.1	−5.0

3. Let $h(x) := \pi/2 - x$, $0 \le x \le \pi$.

   (a) **(10 pts.)** Sketch two periods each of the functions to which the Fourier cosine series (FCS) and Fourier sine series (FSS) converge pointwise. Are either of these series uniformly convergent on $[0,\pi]$? How about on $[\pi/4, 3\pi/4]$? Why?

   (b) **(15 pts.)** Find the FCS for h. Use it to evaluate the sum $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

   (c) **(5 pts.)** On $[0,\pi]$, h has a symmetry property. State it. Use it to explain why all of the even coefficients in the FCS for h vanish.

4. **(20 pts.)** Let σ be real, and not an integer. Find the complex form of the Fourier series for the 2π-periodic function F, where $F(x) = e^{-i\sigma x}$ on $-\pi < x < \pi$. Use this and Parseval's theorem to show that

$$\csc^2(\sigma\pi) = \frac{1}{\pi^2} \sum_{k=-\infty}^{\infty} \frac{1}{(\sigma + k)^2}.$$

5. **(15 pts.)** Do one of the following.

   (a) Sketch a proof for this theorem. *Suppose f is a continuous and 2π-periodic function. Then for each point x, where the derivative of f is defined, the Fourier series of f at x converges to $f(x)$.*

   (b) Recall that the FS for $g(x) = \begin{cases} \pi - x, & 0 \le x \le \pi\,, \\ -\pi - x, & -\pi \le x < 0 \end{cases}$ exhibits the Gibbs phenomenon near $x = 0$. Briefly describe what this is for g, and then show that it is universal.

Integrals

1. $\int udv = uv - \int vdu$
2. $\int \dfrac{dt}{t} = \ln\left|t\right| + C$
3. $\int e^{at}dt = \frac{1}{a}e^{at} + C$
4. $\int t^n e^{at}dt = \frac{1}{a}t^n e^{at} - \frac{n}{a}\int t^{n-1}e^{at}dt$
5. $\int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2}\left(a\cos(bt) + b\sin(bt)\right) + C$
6. $\int e^{at}\sin(bt)dt = \frac{e^{at}}{a^2+b^2}\left(a\sin(bt) - b\cos(bt)\right) + C$
7. $\int t\sin(t)dt = \sin(t) - t\cos(t) + C$
8. $\int t\cos(t)dt = \cos(t) + t\sin(t) + C$
9. $\int \sin(at)dt = -\dfrac{1}{a}\cos(at) + C$
10. $\int \cos(at)dt = \dfrac{1}{a}\sin(at) + C$
11. $\int \tan(at)dt = \dfrac{1}{a}\ln\left|\sec(at)\right| + C$
12. $\int \cot(at)dt = \dfrac{1}{a}\ln\left|\sin(at)\right| + C$
13. $\int \sec(at)dt = \dfrac{1}{a}\ln\left|\sec(at) + \tan(at)\right| + C$
14. $\int \csc(at)dt = \dfrac{1}{a}\ln\left|\csc(at) - \cot(at)\right| + C$
15. $\int \dfrac{dt}{t^2 + a^2} = \dfrac{1}{a}\arctan(t/a) + C$
16. $\int \dfrac{dt}{t^2 - a^2} = \dfrac{1}{2a}\ln\left|\dfrac{t-a}{t+a}\right| + C$