Test I

Instructions: *Show all work in your bluebook. You may use a calculator for numerical computations. You may* not *use a graphing calcular or a calculator that can do symbolics.*

1. Let $g(x) := \pi/2 - x$, $0 \le x \le \pi$.
   (a) **(10 pts.)** Find the FCS for g, and sketch three periods of the function to which it converges pointwise.
   (b) **(5 pts.)** Sketch three periods of the function to which the Fourier *sine* series converges pointwise. (Do *not* compute the coefficients in the series.)
   (c) **(10 pts.)** Define the term *uniform convergence*. Is either series uniformly convergent? If so, which? Why? Will either series exhibit the Gibbs' phenomenon? Briefly explain.

2. **(10 pts.)** Let r be real. Show that the complex form of the Fourier series for the 2π-periodic function F, where $F(x) = e^{rx}$ on $-\pi < x < \pi$, is $F(x) = \frac{e^{\pi r} - e^{-\pi r}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{r - in} e^{inx}$. Use this and Parseval's theorem to sum the series $\sum_{n=-\infty}^{\infty} \frac{1}{r^2+n^2}$.

3. Consider the function $f(t) = e^{-|t|}$. Do the following.
   (a) **(10 pts.)** Show that $\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \frac{1}{1+\lambda^2}$.
   (b) **(5 pts.)** Find the integral $\int_{-\infty}^{\infty} \frac{du}{(1+u^2)^2}$.
   (c) **(5 pts.)** Find the transform $\mathcal{F}[tf(t-1)]$.

4. Let $h(t) = \begin{cases} Ae^{-\alpha t} & t \ge 0\,, \\ 0 & t < 0 \end{cases}$ be the impulse response (IR) for the Butterworth filter $H[f] = h * f$.
   (a) **(10 pts.)** Find $H[f]$, where $f(t) = \begin{cases} 1 & 0 \le t \le 2\,, \\ 0 & t < 0 \text{ or } t > 2 \end{cases}$
   (b) **(5 pts.)** Define the term *causal filter*. Is H causal? Explain.

5. **(15 pts.)** Let $\mathcal{S}_n$ be the space of n periodic sequences. If $y \in \mathcal{S}_n$ and if $z \in \mathcal{S}_n$ is defined by $z_j = y_{j+1}$, show that $\hat{z}_k = w^k \hat{y}_k$, where $w = e^{2\pi i/n}$.

6. **(15 pts.)** Do *one* of the following.
   (a) State and prove the Riemann-Lebesgue Lemma in the case where $f(x)$ is continuously differentiable on the finite interval $[a, b]$.
   (b) State and prove the Sampling Theorem.
   (c) Sketch the proof for the pointwise convergence of the Fourier series a function f that is 2π-periodic, piecewise continuous, and has a piecewise continuous derivative.

Fourier Transform Properties

1. $\hat{f}(\xi) = \mathcal{F}[f](\xi) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx.$
2. $f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\xi) e^{ix\xi} d\xi.$
3. $\mathcal{F}[x^n f(x)](\xi) = i^n \hat{f}^{(n)}(\xi).$
4. $\mathcal{F}[f^{(n)}(x)](\xi) = (i\xi)^n \hat{f}(\xi).$
5. $\mathcal{F}[f(x-a)](\xi) = e^{-i\xi a} \hat{f}(\xi).$
6. $\mathcal{F}[f(bx)](\xi) = \dfrac{1}{b} \hat{f}(\dfrac{\xi}{b}).$
7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\xi) \hat{g}(\xi)$

Integrals

1. $\int u dv = uv - \int v du$
2. $\int \dfrac{dt}{t} = \ln \left| t \right| + C$
3. $\int e^{at} dt = \frac{1}{a} e^{at} + C$
4. $\int t^n e^{at} dt = \frac{1}{a} t^n e^{at} - \frac{n}{a} \int t^{n-1} e^{at} dt$
5. $\int e^{at} \cos(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \cos(bt) + b \sin(bt) \right) + C$
6. $\int e^{at} \sin(bt) dt = \frac{e^{at}}{a^2+b^2} \left(a \sin(bt) - b \cos(bt) \right) + C$
7. $\int t \sin(t) dt = \sin(t) - t \cos(t) + C$
8. $\int t \cos(t) dt = \cos(t) + t \sin(t) + C$
9. $\int \tan(at) dt = \dfrac{1}{a} \ln \left| \sec(at) \right| + C$
10. $\int \cot(at) dt = \dfrac{1}{a} \ln \left| \sin(at) \right| + C$
11. $\int \sec(at) dt = \dfrac{1}{a} \ln \left| \sec(at) + \tan(at) \right| + C$
12. $\int \csc(at) dt = \dfrac{1}{a} \ln \left| \csc(at) - \cot(at) \right| + C$
13. $\int \dfrac{dt}{t^2 + a^2} = \dfrac{1}{a} \arctan(t/a) + C$