Test 1

Instructions: *Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed..*

1. **(25 pts.)** Define the term *uniform convergence*. Let $f(x) = \pi^2 - x^2$, $0 \le x \le \pi$. *Without* finding the series involved, determine whether either the Fourier cosine series for f or the Fourier sine series for f converges uniformly to f on $[0, \pi]$. Give reasons for your answers.

2. **(25 pts.)** Define the term *pointwise convergence*. Let $f(x) = \pi - |x|$, $-\pi \le x \le \pi$. Find the Fourier series for f. Sketch three periods of the function to which this series converges pointwise. Is the convergence uniform? Give a reason for your answer. Use the series you have found to evaluate the series $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$.

3. **(25 pts.)** Define the term *convergence in the mean*. Find the *complex* form of the Fourier series for the function $f(x) = e^x$, $0 \le x \le 2\pi$. Use this series and Parseval's Theorem to evaluate the series $\sum_{n=-\infty}^{\infty} \frac{1}{n^2+1}$.

4. **(25 pts.)** Do *one* of the following. (No extra credit for doing both.)

   (a) You are *given* that the function $f(x) = (\pi^2 x - x^3)/12$, $-\pi \le x \le \pi$ has the Fourier series

   $$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(nx)}{n^3}.$$

   Let $R_N(x) = f(x) - S_N(x)$ denote "residual signal." Show that the "energy" $\|R_N\|^2_{L^2[-\pi,\pi]}$ in the residual signal satisfies

   $$\|R_N\|^2_{L^2[-\pi,\pi]} \le CN^{-5},$$

   where C is some number independent of N.

   (b) Sketch (i.e., outline) a proof for this theorem. *Suppose f is a continuous and 2π-periodic function. Then for each point x, where the derivative of f is defined, the Fourier series of f at x converges to $f(x)$.*