Final Examination

Instructions: *Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed..*

1. **(10 pts.)** Find the Fourier series for $f(\theta) = \begin{cases} 0 & -\pi < \theta \le 0, \\ 1 & 0 < \theta \le \pi. \end{cases}$

2. **(10 pts.)** Let $h(t) = \begin{cases} 2e^{-2t} & t \ge 0\,, \\ 0 & t < 0 \end{cases}$ be the impulse response (IR) for the Butterworth filter $L[f] = h * f$. Find $L[f]$, where
$$f(t) = \begin{cases} 1 & 0 \le t \le 3\,, \\ 0 & t < 0 \text{ or } t > 3\,. \end{cases}$$

3. **(10 pts.)** Let $\mathcal{F}_n$ be the DFT for n-periodic sequences (signals). Find $\hat{a} = \mathcal{F}_4[a]$ if $a = (-2, 1, 0, 1)$. Note: for $n = 4$, $\overline{w} = -i$.

4. **(10 pts.)** Define the term *multiresolution analysis* (MRA). In the case of the Haar MRA, what are V_0, W_0, ϕ, and ψ?

5. **(15 pts.)** For the Haar MRA, $p_0 = p_1 = 1$, and $p_k = 0$ for all other k. Reconstruct the function $f \in V_3$ that has this Haar wavelet decomposition:
$$a^1 = [3/2, -1] \quad b^1 = [-1, -3/2] \quad b^2 = [-3/2, -3/2, -1/2, -1/2],$$
where the first entry in each list corresponds to $k = 0$, the second to $k = 1$, and so on.

6. The scaling relation for an MRA is $\phi(x) = \sum_{k=-\infty}^{\infty} p_k \phi(2x - k)$.
   (a) **(10 pts.)** Draw the corresponding decomposition and reconstruction diagrams. Define the impulse response functions for the high pass and low pass filters, as well as the operators $2\uparrow$ and $2\downarrow$.
   (b) **(10 pts.)** Show that $\hat{\phi}(\xi) = P(e^{-i\xi/2})\hat{\phi}(\xi/2)$, where $P(z) = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_k z^k$. State the conditions that $P(z)$ satisfies.
   (c) **(10 pts.)** Explain how the Daubechies wavelets are classified in terms of $P(z)$. What is the connection with "vanishing moments"?

7. **(15 pts.)** Let ϕ be a scaling function with compact support and satisfying $\int \overline{\phi(x)}dx = 1$. Show that for a continuous signal f the highest level coefficients satisfy $a_k^j \approx f(2^{-j}k)$.