Test 2

Instructions: *Show all work in your bluebook, except for problem 1. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed.*

1. Place your answer in the space provided.

   (a) **(5 pts.)** Define the terms *linear, time-invariant filter* and a *causal filter.*

   (b) **(5 pts.)** A signal has length $n = 2^{10}$. Approximately how many multiplications are required to compute the DFT of the signal using the FFT algorithm? How many are needed when using the matrix multiplication method?

   (c) **(5 pts.)** You are given that for $f(t) = e^{-|t|}$, $\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}}\,(1+\lambda^2)^{-1}$. Use the properties in the table to find $\mathcal{F}[te^{-|t|}]$.

   (d) **(5 pts.)** Let $f(x) = \pi - x$, $0 \le x \le \pi$. Sketch two periods of the pointwise limit of its Fourier cosine series (FCS). Is the FCS uniformly convergent? Why or why not?

2. **(20 pts.)** You are *given* that, on $-\pi \le x \le \pi$,

$$|x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos((2n-1)x)}{(2n-1)^2}. \tag{1}$$

Use (1) and Parseval's equation to find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4}$.

3. Consider the filter $L[f] = f * h$, where $h(t) := \begin{cases} 1/d & 0 \le t \le d \\ 0 & \text{otherwise.} \end{cases}$

   (a) **(10 pts.)** Show that $L[f] = d^{-1} \int_{t-d}^{t} f(\tau) d\tau$

   (b) **(10 pts.)** Find the system function, $\hat{h}$, and the transformed version of the filter, $\widehat{L[f]}(\lambda)$.

4. **(10 pts.)** Derive this property of the Fourier transform: $\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda)$.

5. **(10 pts.)** Find the discrete Fourier transform $\hat{y} = \mathcal{F}_4[y]$ if $y = (1, -1, 0, 1)$.

6. **(20 pts.)** Do *one* of the following. (No extra credit for doing more.)

   (a) State and prove the convolution theorem for the DFT.

   (b) State the Sampling Theorem and sketch a proof of it.

   (c) Show that if a linear filter L is causal, the impulse response function $h(t)$ is 0 for all $t < 0$.

Integrals

1. $\int udv = uv - \int vdu$
2. $\int \frac{dt}{t} = \ln\left|t\right| + C$
3. $\int e^{at}dt = \frac{1}{a}e^{at} + C$
4. $\int t^n e^{at}dt = \frac{1}{a}t^n e^{at} - \frac{n}{a}\int t^{n-1}e^{at}dt$
5. $\int e^{at}\cos(bt)dt = \frac{e^{at}}{a^2+b^2}\left(a\cos(bt) + b\sin(bt)\right) + C$
6. $\int e^{at}\sin(bt)dt = \frac{e^{at}}{a^2+b^2}\left(a\sin(bt) - b\cos(bt)\right) + C$
7. $\int t\sin(t)dt = \sin(t) - t\cos(t) + C$
8. $\int t\cos(t)dt = \cos(t) + t\sin(t) + C$
9. $\int \sin(at)dt = -\frac{1}{a}\cos(at) + C$
10. $\int \cos(at)dt = \frac{1}{a}\sin(at) + C$
11. $\int \tan(at)dt = \frac{1}{a}\ln\left|\sec(at)\right| + C$
12. $\int \cot(at)dt = \frac{1}{a}\ln\left|\sin(at)\right| + C$
13. $\int \sec(at)dt = \frac{1}{a}\ln\left|\sec(at) + \tan(at)\right| + C$
14. $\int \csc(at)dt = \frac{1}{a}\ln\left|\csc(at) - \cot(at)\right| + C$
15. $\int \frac{dt}{t^2 + a^2} = \frac{1}{a}\arctan(t/a) + C$
16. $\int \frac{dt}{t^2 - a^2} = \frac{1}{2a}\ln\left|\frac{t-a}{t+a}\right| + C$

Properties of the Fourier Transform

1. $\hat{f}(\lambda) = \mathcal{F}[f](\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\lambda} dx.$

2. $f(t) = \mathcal{F}^{-1}[\hat{f}](t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\lambda) e^{it\lambda} d\lambda.$

3. $\mathcal{F}[t^n f(t)](\lambda) = i^n \hat{f}^{(n)}(\lambda).$

4. $\mathcal{F}[f^{(n)}(t)](\lambda) = (i\lambda)^n \hat{f}(\lambda).$

5. $\mathcal{F}[f(t-a)](\lambda) = e^{-i\lambda a} \hat{f}(\lambda).$

6. $\mathcal{F}[f(bt)](\lambda) = \frac{1}{b} \hat{f}(\frac{\lambda}{b}).$

7. $\mathcal{F}[f * g] = \sqrt{2\pi} \hat{f}(\lambda) \hat{g}(\lambda)$

8. $\text{sinc}(t) := \frac{\sin(\pi t)}{\pi t} = \mathcal{F}^{-1}[\chi_\pi]$, where $\chi_\pi(\lambda) = \begin{cases} 1/\sqrt{2\pi}, & -\pi \leq \lambda \leq \pi \\ 0, & |\lambda| > \pi \end{cases}$.