Test 2

Instructions: *Show all work in your bluebook. Cell phones, laptops, calculators that do linear algebra or calculus, and other such devices are* not *allowed.*

1. State the theorem and the definitions below.
   (a) **(10 pts.)** *Sampling Theorem* and *Nyquist frequency.*
   (b) **(5 pts.)** *Z-transform*

2. **(20 pts.)** Consider the filter $L[f] = f * h$, where $h(t)$ is given below. Find the system function, $\hat{h}$, and the Fourier transform, $\widehat{L[f]}(\lambda)$. Is L causal? Explain. (You *don't* need to find $f * h$ here.)

$$h(t) := \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

3. Let $\mathcal{F}_n$ be the DFT for n-periodic sequences (signals).
   (a) **(5 pts.)** If a signal has 2^{10} samples, how many operations does it take to find the DFT using the FFT algorithm? How many via the matrix multiplication method?
   (b) **(10 pts.)** Find $\mathcal{F}_4[a]$ if $a = (1, -2, 1, 3)$. Note: for $n = 4$, $\overline{w} = -i$.
   (c) **(10 pts.)** Explain how you would use the FFT to approximate the $\widehat{f}(\lambda)$, given samples $f(kT)$, $k = 0, \ldots, 2^n - 1$? (I'm not asking for formulas here, just a description of the process.)

4. **(25 pts.)** For the Haar MRA, the decomposition formulas are given by $a^j_k = (a^{j+1}_{2k} + a^{j+1}_{2k+1})/2$ and $b^j_k = (a^{j+1}_{2k} - a^{j+1}_{2k+1})/2$. Given that $L[x]_k = \frac{1}{2}(x_k + x_{k+1})$ and $H[x]_k = \frac{1}{2}(x_k - x_{k+1})$, find the impulse responses ℓ and h for L and H, respectively. What is $2{\downarrow}$? Why is it needed in Figure 1?

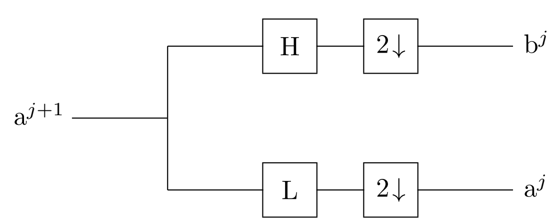

Figure 1: Haar decomposition diagram.

Please turn over

5. **(15 pts.)** Do *one* of the following. (No extra credit for doing more.)

   (a) State and prove the convolution theorem for the DFT.

   (b) Derive the decomposition formulas in problem 4, starting with $a_k^j = 2^j \int_{-\infty}^{\infty} f(x)\phi(2^j x - k)dx$ and $b_k^j = 2^j \int_{-\infty}^{\infty} f(x)\psi(2^j x - k)dx$.

Fourier Transform Properties

1. $\hat{f}(\lambda) = \mathcal{F}[f](\lambda) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)e^{-ix\lambda}dx.$
2. $\dfrac{f(x^+) + f(x^-)}{2} = \mathcal{F}^{-1}[\hat{f}](x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\lambda)e^{ix\lambda}d\lambda.$
3. $\mathcal{F}[x^n f(x)](\lambda) = i^n \hat{f}^{(n)}(\lambda).$
4. $\mathcal{F}[f^{(n)}(x)](\lambda) = (i\lambda)^n \hat{f}(\lambda).$
5. $\mathcal{F}[f(x-a)](\lambda) = e^{-i\lambda a}\hat{f}(\lambda).$
6. $\mathcal{F}[f(bx)](\lambda) = \dfrac{1}{b}\hat{f}(\dfrac{\lambda}{b}).$
7. $\mathcal{F}[f * g] = \sqrt{2\pi}\hat{f}(\lambda)\hat{g}(\lambda)$ and $\mathcal{F}^{-1}[\hat{f}(\lambda)\hat{g}(\lambda)] = \dfrac{1}{\sqrt{2\pi}} f * g$

Integrals

1. $\int e^{at}dt = \dfrac{1}{a}e^{at} + C$
2. $\int t^n e^{at}dt = \dfrac{1}{a}t^n e^{at} - \dfrac{n}{a}\int t^{n-1}e^{at}dt$
3. $\int t\sin(t)dt = \sin(t) - t\cos(t) + C$
4. $\int t\cos(t)dt = \cos(t) + t\sin(t) + C$
5. $\int e^{at}\cos(bt)dt = \dfrac{e^{at}}{a^2+b^2}\left(a\cos(bt) + b\sin(bt)\right) + C$
6. $\int e^{at}\sin(bt)dt = \dfrac{e^{at}}{a^2+b^2}\left(a\sin(bt) - b\cos(bt)\right) + C$
7. $\int \cos(at)\cos(bt)dt = \dfrac{\sin((a+b)t)}{2(a+b)} + \dfrac{\sin((a-b)t)}{2(a-b)} + C,\ a \neq b$
8. $\int \sin(at)\sin(bt)dt = \dfrac{\sin((a+b)t)}{2(a+b)} - \dfrac{\sin((a-b)t)}{2(a-b)} + C,\ a \neq b$
9. $\int \sin(at)\cos(bt)dt = -\dfrac{\cos((a+b)t)}{2(a+b)} - \dfrac{\cos((a-b)t)}{2(a-b)} + C,\ a \neq b$