Quasi-interpolation in shift invariant spaces

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Abstract

Let $s \geq 1$ be an integer, $\phi : \mathbb{R}^s \to \mathbb{R}$ be a compactly supported function, and $S(\phi)$ denote the linear span of $\{\phi(\cdot - \mathbf{k}) \ : \ \mathbf{k} \in \mathbb{Z}^s\}$. We consider the problem of approximating a continuous function $f : \mathbb{R}^s \to \mathbb{R}$ on compact subsets of $\mathbb{R}^s$ from the classes $S(\phi(h\cdot))$, $h > 0$, based on samples of the function at scattered sites in $\mathbb{R}^s$. We demonstrate how classical polynomial inequalities lead to the construction of local, quasi-interpolatory operators for this purpose.

1 Introduction

An important subject in approximation theory is the study of shift-invariant spaces generated by a fixed function. If $s \geq 1$ is an integer, and $\phi : \mathbb{R}^s \to \mathbb{R}$ is a compactly supported function, the *shift-invariant space generated by* ϕ is defined by

$$S(\phi) := \text{span}\{\phi(\cdot - \mathbf{k}) \ : \ \mathbf{k} \in \mathbb{Z}^s\}. \tag{1.1}$$

A primary example of the *generator* ϕ is a cube spline. Other important examples include radial basis functions. The approximation capability of shift-invariant spaces has been studied by de Boor, DeVore and Ron [2]. As far as the actual construction of the approximation is concerned, in much of the literature, one constructs and studies functions from

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$S(\phi)$ that interpolate the data $\{(\mathbf{k}, f(\mathbf{k}))\}_{\mathbf{k}\in\mathbb{Z}^s}$ for some function $f : \mathbb{R}^s \to \mathbb{R}$ (e.g., [7, 4]). However, such constructions are necessarily computationally expensive, since each data is a constraint, forcing the number of parameters in the approximant to increase with the amount of data. In particular, interpolatory approximation typically does not yield the best order of approximation. Also, when the data is noisy, an exact interpolation is unlikely to yield a model that accurately reflects the actual relationship that generates the data.

For these reasons, one considers quasi-interpolation operators, which are based on the data, but are not interpolatory. Instead, the operators are required to reproduce polynomials of a certain fixed degree. In the case of univariate spline functions, such operators were studied in [8, 3]. A more or less comprehensive treatment of these operators is given by Chui and Diamond in [1]. As one of the applications of their theory, they point out the construction of quasi-linear operators based on "semi-scattered" data, based on solving a certain system of equations. We observe that some conditions are necessary on the location of the data points in order that the system of equations be solvable.

In this paper, we describe a construction of quasi-linear operators based on scattered data, with no conditions on their location. The degree of polynomials which are preserved by our operators will depend upon the density of the data. Rather than solving a system of equations, we will need to solve a quadratic (or linear) programming problem of a size having the same order of magnitude. Our constructions are novel applications of the classical Markov inequalities for polynomials.

In the next section, we recall certain notations and results from the paper [1]. Our construction is described in Section 3.

2 Preliminaries

In this section, we recall certain notations, terminology, and results from the paper [1] of Chui and Diamond.

Let $s \geq 1$ be an integer, $\phi : \mathbb{R}^s \to \mathbb{R}$ be a compactly supported function, and for $h > 0$, let $\phi_h(\mathbf{x}) := \phi(h\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^s$. We say that a bounded linear operator L, whose domain and range are function spaces on $\mathbb{R}^s$, is a *local linear operator,* if there is compact subset $K \subset \mathbb{R}^s$ such that $Lf(\mathbf{x}) = 0$ for all f vanishing on $\mathbf{x} + K$, $\mathbf{x} \in \mathbb{R}^s$. A local linear functional is defined analogously. The symbol $\Pi^T_{m,s}$ denotes the class of all polynomials in s variables of total degree at most m. The class of all $m + 1$ times continuously differentiable functions from $\mathbb{R}^s$ to $\mathbb{R}$ is denoted by C^{m+1}.

A linear operator $Q : C^{m+1} \to S(\phi)$ is called a *quasi-interpolatory operator of order* m if Q is local, and if $Q(P) = P$ for all $P \in \Pi^T_{m,s}$. The symbol QL_m denotes the set of all quasi-interpolation operators H of order m that have the form

$$H(f; \mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^s} \lambda(f(\cdot + \mathbf{k}))\phi(\mathbf{x} - \mathbf{k}) \tag{2.1}$$

for some local linear functional λ. The class of all local linear functionals λ such that the operator defined by (2.1) is in QL_m will be denoted by Λ_m. It is easy to check that if $f \in C^{m+1}$ is a compactly supported, $m+1$ times continuously differentiable function, and Q is a quasi-interpolatory operator of order m, then

$$\max_{\mathbf{x}\in\mathbb{R}^s}\left|f(\mathbf{x}) - Q\Big(f(h\cdot);\mathbf{x}/h\Big)\right| = \mathcal{O}(h^{m+1}), \qquad h \to 0+\,.$$

In [1], Chui and Diamond have proved the following interesting theorem and discussed various explicit constructions and applications.

Theorem 2.1 *Let $m \geq 0$ be an integer, and $\phi : \mathbb{R}^s \to \mathbb{R}$ be a compactly supported function. We assume further that $\Pi^T_{m,s} \subset S(\phi)$ and that*

$$\sum_{\mathbf{k}\in\mathbb{Z}^s} \phi(\mathbf{x}-\mathbf{k}) = 1, \qquad \mathbf{x} \in \mathbb{R}^s. \tag{2.2}$$

(a) *If $\lambda \in \Lambda_m$ then $\lambda_1 \in \Lambda_m$ if and only if $\lambda_1(P) = \lambda(P)$ for all $P \in \Pi^T_{m,s}$, and λ_1 is a local linear functional.*
(b) *If $\lambda_{\mathbf{k}} \in \Lambda_m$, $\mathbf{k} \in \mathbb{Z}^s$ then the operator*

$$f \mapsto \sum_{\mathbf{k}\in\mathbb{Z}^s} \lambda_{\mathbf{k}}(f(\cdot+\mathbf{k}))\phi(\mathbf{x}-\mathbf{k}),$$

if it is a local linear operator, is a quasi-interpolatory operator of order m.

To complete the discussion, we review the construction of a local functional γ such that

$$P(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^s} \gamma(P(\cdot+\mathbf{k}))\phi(\mathbf{x}-\mathbf{k}), \qquad P \in \Pi^T_{m,s}. \tag{2.3}$$

Since $P \in S(\phi)$, there exist constants $c_{\mathbf{j}}$ such that

$$P(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^s} c_{\mathbf{j}}\phi(\mathbf{x}-\mathbf{j}), \qquad \mathbf{x} \in \mathbb{R}^s. \tag{2.4}$$

If A is the infinite block diagonal matrix $[\phi(\mathbf{k}-\mathbf{j})]$, $\mathbf{P}$ denotes the vector $(P(\mathbf{k}))$, $\mathbf{c}$ denotes the vector $(c_{\mathbf{j}})$ then (2.4) with $\mathbf{x} = \mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^s$ yields the system of equations

$$\mathbf{P} = A\mathbf{c}. \tag{2.5}$$

We observe that in view of (2.2), $\sum_{\mathbf{j}\in\mathbb{Z}^s}\phi(\mathbf{j}) = 1$. Hence, with I denoting the (infinite) identitiy matrix, we get

$$((I-A)\mathbf{P})(\mathbf{k}) = P(\mathbf{k}) - \sum_{\mathbf{j}\in\mathbb{Z}^s}\phi(\mathbf{j})P(\mathbf{k}-\mathbf{j}) = \sum_{\mathbf{j}\in\mathbb{Z}^s}\phi(\mathbf{j})\left(P(\mathbf{k}) - P(\mathbf{k}-\mathbf{j})\right).$$

Since ϕ is compactly supported, the last sum is actually a finite sum. Further, the quantity $P(\mathbf{k}) - P(\mathbf{k}-\mathbf{j})$ is a polynomial in $\mathbf{k}$ of degree not exceeding $m-1$. Thus, the quantity $(\sum_{\ell=0}^m (I-A)^\ell\mathbf{P})(\mathbf{j})$ can be calculated for each $\mathbf{j}$ with only finitely many computations.

In view of the "degree reducing" property, it can be verified easily that the solution of the matrix equation (2.5) is given by

$$\mathbf{c} = \sum_{\ell=0}^{m} (I - A)^{\ell} \mathbf{P}. \tag{2.6}$$

The matrix $\sum_{\ell=0}^{m}(I-A)^{\ell}$ is of the form $[\psi(\mathbf{j}-\mathbf{k})]$ for some compactly supported function ψ. We define

$$\gamma(f) = \sum_{\mathbf{j}\in\mathbb{Z}^s} \psi(\mathbf{j}) f(-\mathbf{j}). \tag{2.7}$$

Since ψ is compactly supported, γ is a local functional. The identity (2.3) follows from (2.6) and (2.4).

We note that the operator $f \mapsto \sum_{\mathbf{k}\in\mathbb{Z}^s} \gamma(f(\cdot+\mathbf{k}))\phi(\cdot-\mathbf{k})$ is in QL_m, and each coefficient $\gamma(f(\cdot+\mathbf{k}))$ is evaluated using the values of f at lattice points in a neighborhood of $\mathbf{k}$. Our aim in this paper is to provide a construction of an operator as in part (d) of the above theorem, where the functionals $\lambda_{\mathbf{k}}$ are evaluated using the values of f at scattered sites in $\mathbf{k}+[-1,1]^s$. We need to consider only the case $\mathbf{k}=\mathbf{0}$.

3 A local functional based on scattered data

Let $\mathcal{C}_0$ be a set of distinct points in $[-1,1]^s$, with the mesh norm defined by

$$\delta_{\mathcal{C}_0} := \max_{\mathbf{x}\in[-1,1]^s} \mathrm{dist}(\mathbf{x}, \mathcal{C}_0) := \max_{\mathbf{x}\in[-1,1]^s} \min_{\mathbf{y}\in\mathcal{C}_0} \mathrm{dist}(\mathbf{x},\mathbf{y}), \tag{3.1}$$

where $\mathrm{dist}(\mathbf{x},\mathbf{y})$ denotes the ℓ^∞ distance between the points $\mathbf{x}$ and $\mathbf{y}$ in $[-1,1]^s$. Given the data $\{(\xi, f(\xi))\}_{\xi\in\mathcal{C}_0}$, we wish to construct a functional of the form $f \mapsto \sum_{\xi\in\mathcal{C}_0} a_\xi f(\xi)$, that is in Λ_m.

Towards this end, we first assume a certain reduction of the data. We divide $[-1,1]^s$ into congruent subcubes of side not exceeding $2\delta_{\mathcal{C}_0}$. Each of these subcubes has at least one point of $\mathcal{C}_0$, and we may choose a subset $\mathcal{C}$, so that each subcube has exactly one point of $\mathcal{C}_0$. The collection of the subcubes is denoted by $\mathcal{R}_{\mathcal{C}} := \{R_\xi \ : \ \xi\in\mathcal{C}\}$. Also, it is easy to verify that $\delta_{\mathcal{C}_0} \le \delta_{\mathcal{C}} \le 2\delta_{\mathcal{C}_0}$. In the following discussion, we will utilize only the subset $\mathcal{C}$. In the remainder of this section, m is fixed, and all the constants may depend upon m and s. In the sequel, $c, c_1, \cdots$ will denote positive constants, possibly depending on m and s. Their value may be different at different ocurrences, even within the same formula.

Our construction is based on the following theorem.

Theorem 3.1 *Let $\mathcal{C}$, $\mathcal{R}_{\mathcal{C}}$ be as above, and γ be defined as in (2.7). There exists a constant $\eta := \eta(m,s)$ with the following property. If $\delta_{\mathcal{C}} \le \eta$, then there exist real numbers $\{a_\xi \ : \ \xi\in\mathcal{C}\}$, such that*

$$|a_\xi| \le c\,\mathrm{vol}_s(R_\xi) \le c\delta_{\mathcal{C}}^s, \qquad \xi\in\mathcal{C}, \tag{3.2}$$

and

$$\sum_{\xi\in\mathcal{C}} a_\xi P(\xi) = \gamma(P), \qquad P\in\Pi_{m,s}^T. \tag{3.3}$$

Theorem 3.1 will be proved as consequence of a very general theorem which we proved in [6] (Proposition 3.1 below). To describe this theorem, we first introduce the notion of a norming set. Let X be a finite dimensional vector space, with norm $\|\cdot\|_X$, and let $\mathcal{Z}\subset X^*$ be a finite set.

Definition 3.1 *We will say that $\mathcal{Z}$ is a* norming set *for X if the mapping $T_{\mathcal{Z}}: X\to\mathbb{R}^{|\mathcal{Z}|}$ defined by $T_{\mathcal{Z}}(x)=(z(x))_{z\in\mathcal{Z}}$ is injective. We will call $T_{\mathcal{Z}}$ the* sampling operator.

We remark that if $\mathcal{V}:=T_{\mathcal{Z}}(X)$ is the range of $T_{\mathcal{Z}}$, then the injectivity of $T_{\mathcal{Z}}$ implies that $T_{\mathcal{Z}}^{-1}:\mathcal{V}\to X$ exists. Let $\mathbb{R}^{|\mathcal{Z}|}$ have a norm $\|\cdot\|_{\mathbb{R}^{|\mathcal{Z}|}}$, with $\|\cdot\|_{\mathbb{R}^{|\mathcal{Z}|^*}}$ being its dual norm on $\mathbb{R}^{|\mathcal{Z}|^*}$. Equip $\mathcal{V}$ with the induced norm, and let $\|T_{\mathcal{Z}}^{-1}\|:=\|T_{\mathcal{Z}}^{-1}\|_{\mathcal{V}\to X}$. In [6], we proved the following:

Proposition 3.1 *Let $\mathcal{Z}$ be a norming set for X, with $T_{\mathcal{Z}}$ being the corresponding sampling operator. If $y\in X^*$, with $\|y\|_{X^*}\le 1$, then there exist real numbers $\{a_z\}_{z\in\mathcal{Z}}$, depending only on y such that for every $x\in X$,*

$$y(x)=\sum_{z\in\mathcal{Z}}a_z z(x), \tag{3.4}$$

and

$$\|(a_z)\|_{\mathbb{R}^{|\mathcal{Z}|^*}}\le\|T_{\mathcal{Z}}^{-1}\|. \tag{3.5}$$

We shall show that if $\delta_{\mathcal{C}}$ is sufficiently small, then the class of evaluation functionals at points of $\mathcal{C}$ is a norming set for $\Pi_{m,s}^T$ equipped with the L^1 norm and $\mathbb{R}^s$ with an ℓ^1 norm. Theorem 3.1 then follows by taking γ to be the functional y in Proposition 3.1.

Theorem 3.2 *Let $\mathcal{C}$, $\mathcal{R}_{\mathcal{C}}$ be as above. There exists a constant $\eta:=\eta(m,s)$ such that if $\delta_{\mathcal{C}}\le\eta$, then for any $P\in\Pi_{m,s}^T$,*

$$(1-\eta)\sum_{\xi\in\mathcal{C}}\mathrm{vol}_s(R_\xi)|P(\xi)|\le\int_{[-1,1]^s}|P(\mathbf{x})|d\mathbf{x}\le(1+\eta)\sum_{\xi\in\mathcal{C}}\mathrm{vol}_s(R_\xi)|P(\xi)|. \tag{3.6}$$

PROOF. Let $P\in\Pi_{m,s}^T$. The estimate (3.6) follows easily from the following inequality:

$$\sum_{\xi\in\mathcal{C}}\int_{R_\xi}|P(\mathbf{x})-P(\xi)|d\mathbf{x}\le c\delta_{\mathcal{C}}\int_{[-1,1]^s}|P(\mathbf{x})|d\mathbf{x}. \tag{3.7}$$

Let $\xi\in\mathcal{C}$, and δ be the length of one side of R_ξ. We write P_j for the partial derivative of P with respect to its j-th variable. Using the Nikolskii inequality [9, Section 4.9.6(36)], we deduce that for any j, $1\le j\le s$,

$$\max_{\mathbf{t}\in\mathcal{R}_\xi}|P_j(\mathbf{t})|\le\left(\frac{4m^2}{\delta}\right)^s\int_{R_\xi}|P_j(\mathbf{t})|d\mathbf{t}.$$

Hence, for $\mathbf{x}\in R_\xi$, we have

$$|P(\mathbf{x})-P(\xi)|\le\delta\sum_{j=1}^s\max_{\mathbf{t}\in\mathcal{R}_\xi}|P_j(\mathbf{t})|\le\delta\left(\frac{4m^2}{\delta}\right)^s\sum_{j=1}^s\int_{R_\xi}|P_j(\mathbf{t})|d\mathbf{t}.$$

Therefore,

$$\sum_{\xi\in\mathcal{C}} \int_{R_\xi} |P(\mathbf{x}) - P(\xi)|d\mathbf{x} \le \delta(4m^2)^s \sum_{j=1}^{s} \int_{[-1,1]^s} |P_j(\mathbf{t})|d\mathbf{t}.$$

The estimate (3.7) now follows from the Markov inequality [5]. □

We summarize by giving an algorithm for the construction of the local linear functionals involved.

ALGORITHM.

Given: A set $\mathcal{C}_0$ of points in $[-1,1]^s$ with $\delta_{\mathcal{C}_0} \le \eta(m,s)/2$, where $\eta(m,s)$ is as in Theorem 3.1, and the functional γ as in (2.7).

Objective: To find real numbers $\{a_\xi\}_{\xi\in\mathcal{C}_0}$ such that the functional $\lambda(f) := \sum_{\xi\in\mathcal{C}_0} a_\xi f(\xi)$ satisfies

$$\lambda(P) = \gamma(P), \qquad P \in \Pi^T_{m,s}.$$

Steps:

1. Divide $[-1,1]^s$ into congruent subcubes of side not exceeding $2\delta_{\mathcal{C}_0}$.
2. Choose $\mathcal{C} \subseteq \mathcal{C}_0$, so that each subcube has exactly one point of $\mathcal{C}$.
3. Solve the following (underdetermined) system of equations for the unknowns a_ξ, $\xi \in \mathcal{C}$.

$$\sum_{\xi\in\mathcal{C}} a_\xi \xi^{\mathbf{k}} = \gamma((\cdot)^{\mathbf{k}}), \qquad |\mathbf{k}| \le m. \tag{3.8}$$

4. Set $a_\xi := 0$ if $\xi \in \mathcal{C}_0 \setminus \mathcal{C}$.
5. Output $(a_\xi)_{\xi\in\mathcal{C}_0}$.

REMARK. Theorem 3.1 asserts the existence of a solution to (3.8). One way to solve the system (3.8) is to solve the quadratic problem

Minimize $\sum_{\xi\in\mathcal{C}} a_\xi^2$ *subject to the constraints (3.8).*

This amounts to solving the following system of equations for the unknowns $\mu_{\mathbf{k}}$, $|\mathbf{k}| \le m$, where $\gamma_{\mathbf{k}} := \gamma((\cdot)^{\mathbf{k}})$.

$$\sum_{|\mathbf{j}|\le m, \mathbf{j}\in\mathbb{Z}^s} \mu_{\mathbf{j}} \sum_{\xi\in\mathcal{C}} \xi^{\mathbf{k}+\mathbf{j}} = \gamma_{\mathbf{k}} + \sum_{|\mathbf{j}|\le m, \mathbf{j}\in\mathbb{Z}^s} \gamma_{\mathbf{j}} \sum_{\xi\in\mathcal{C}} \xi^{\mathbf{k}}. \tag{3.9}$$

One then sets

$$a_\xi := \sum_{|\mathbf{j}|\le m, \mathbf{j}\in\mathbb{Z}^s} (\mu_{\mathbf{j}} \xi^{\mathbf{j}} - \gamma_{\mathbf{j}}), \qquad \xi \in \mathcal{C}.$$

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