Nonstationary Wavelets on the m-Sphere for Scattered Data

by

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Abstract

We construct classes of nonstationary wavelets generated by what we call *spherical basis functions* (SBFs), which comprise a subclass of Schoenberg's positive definite functions on the m-sphere. The wavelets are intrinsically defined on the m-sphere, and are independent of the choice of coordinate system. In addition, they may be orthogonalized easily, if desired. We will discuss decomposition, reconstruction, and localization for these wavelets. In the special case of the 2-sphere, we derive an uncertainty principle that expresses the trade-off between localization and the presence of high harmonics—or high frequencies—in expansions in spherical harmonics. We discuss the application of this principle to the wavelets that we construct.

I. Introduction

Geophyiscal or meteorological data collected over the surface of the earth via satellites or ground stations will invariably come from scattered sites. Synthesizing and analyzing such data is the motivation for the work that is presented here. In this paper, we investigate a nonstationary multiresolution analysis not only for $\mathbb{S}^2$, but also for the m-sphere, $\mathbb{S}^m$ when $m \geq 2$. The sampling spaces that it employs are

$$\mathcal{V}_N = \mathrm{Span}\{G(p \cdot \xi_j)\}_{j=1}^N$$

where the ξ_j's are scattered points on $\mathbb{S}^m$ and G is a function from a subclass of Schoenberg's positive definite functions on $\mathbb{S}^m$ [24]. The subclass is described in detail in §II. Functions in it behave in many ways like radial basis functions [21], and they will be called *spherical basis functions* (SBFs).

Our method for constructing sampling spaces and wavelets on the m-sphere is intrinsic. The argument in $G(p \cdot \xi_j)$ is actually $p \cdot \xi_j = \cos(\theta(p, \xi_j))$, where $\theta(p, \xi_j)$ is the geodesic distance from p to ξ_j on $\mathbb{S}^m$. It is thus a geometric quantity that is independent of the choice of coordinates, as are the sampling spaces and the wavelet spaces that we will construct in §III.

The intrinsic nature of these wavelets and sampling spaces makes them attractive for the applications mentioned earlier. Because they are defined in a way that is independent of particular choices of coordinates, they avoid the artificial boundaries, distortions, and singularities inherent in the other approaches to analyzing data on spheres. Another reason for the attractiveness of the wavelets and sampling spaces discussed here is that they are naturally applicable to fitting surfaces to data that is *scattered*. We will further discuss the ability of SBFs to interpolate in §II. Here, we point out that this feature of the theory is especially important in view of there being no analog on the 2-sphere to the infinite, scaled lattices that are so central to the construction of the wavelets in Euclidean spaces. Indeed, only the regularly spaced vertices of the platonic solids form such lattices. Unfortunately, the solid with the largest vertex set is the dodecahedron, which has twenty vertices. This means that on $\mathbb{S}^2$ every set with twenty-one or more points is scattered! We hasten to point out that the approach taken here is not limited to the 2-sphere, although that may be where most applications lie.

Different approaches to creating wavelets on the 2-sphere have recently been taken by Freeden and Windheuser [11] and by Tasche [26]. In [11], a continuous wavelet transform on the sphere is discretized. In [26], one first maps the rectangle $[0, \pi] \times [0, 2\pi]$ (or "box" in higher dimensions) to the sphere via standard spherical coordinates, and then one constructs wavelets by taking tensor products of Euclidean wavelets designed for intervals [5, 6] with periodic wavelets [4, 14, 18, 20, 22]. Of course, doing this results in singularities and distortions near the poles, and these must be dealt with.

The mathematical motivation for the work presented here stems from work by several researchers. Chui, Jetter, Stöckler, and Ward [3] employ radial basis functions to construct wavelets that deal with scattered data in Euclidean spaces. Xu and Cheney [30] gave general conditions on positive definite spherical functions for such functions to have invertible interpolation matrices–that is, for them to be useful for solving scattered data interpolation problems. Narcowich and Ward [18] constructed periodic wavelets out of the periodic version of an SBF, and in particular were successful in generating a multiresolution analysis where the corresponding wavelets, which based on translations only, possessed many of the wavelet properties one finds desirable—orthogonality, localization, small uncertainty product, and so on.

The outline of the paper goes as follows. In §II, after reviewing various aspects of positive definite functions on $\mathbb{S}^m$, we introduce SBFs, and develop some of their useful properties, including explicit formulas for certain Grammian matrices that arise in the work. In particular, we give several examples of SBFs, and even construct an SBF that is "locally supported" and therefore gives rise to banded interpolation matrices. In §III, we explore the properties of the sampling spaces and wavelet spaces associated with SBFs. Moreover, we establish an $L^2(\mathbb{S}^m)$ density result for the sampling spaces, and provide decomposition and reconstruction formulas. Finally, we show that a (nonorthogonal) basis for the wavelet space comprises certain columns in the inverse of a Grammian matrix, and we use this result to construct all possible orthogonal wavelets at a given level. In §IV, we provide criteria for measuring the localization of wavelets, and exhibit a family of wavelets with good localization an "uncertainty principle" for $\mathbb{S}^2$. This uncertainty principle measures the trade-off between localization on $\mathbb{S}^2$ and the presence of high harmonics—or high frequencies—in expansions in spherical harmonics. We discuss the application of this principle to the wavelets that we construct. If one were doing quantum mechanics on $\mathbb{S}^2$, the uncertainty principle that we derive would embody a trade-off between position on $\mathbb{S}^m$ and total angular momentum.

A partial summary of the results that we obtain here was presented in [19].

II. Spherical Basis Functions

To construct the sampling spaces, we will employ positive definite functions defined on $\mathbb{S}^m$, the unit sphere embedded in $\mathbb{R}^{m+1}$. These functions, which Schoenberg introduced in [24], are defined as follows. Let $p \cdot q$ be the usual "dot" product in $\mathbb{R}^{m+1}$, and let $\theta(p,q) = \arccos(p \cdot q)$ be the geodesic distance between two points p and q on $\mathbb{S}^m$. Thus, $\theta(p,q)$ is the shorter arc of the great circle joining the two points p and q. A real-valued, continuous function of $\theta(p,q)$ is said to be a positive definite function on $\mathbb{S}^m$ if for every finite set of points the associated (self-adjoint) interpolation matrix is nonnegative. In [24], Schoenberg completely characterized all such functions in terms of expansions in ultraspherical (Gegenbauer) polynomials. Because the Legendre polynomials are somewhat easier to use than the ultraspherical polynomials, and are, up to constant multiples, the same set, we will use them to give Schoenberg's representation theorem [24, eq. (1.5)]:

$$G(\cos\theta(p,q)) = G(p \cdot q) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(m+1; p \cdot q), \tag{2.1}$$

where the $a_\ell \geq 0$ and satisfy $\sum_{\ell=0}^{\infty} a_\ell < \infty$. The function $P_\ell(m+1;x)$ [12, p. 176; 16, p. 21] is the *Legendre polynomial of degree ℓ in $m+1$ dimensions*, and it is normalized so that $P_\ell(m+1;1) = 1$.

Special positive definite spherical functions have long been used in connection with spherical spline interpolation [9, 27]. However, it was only recently that Xu and Cheney

[30] gave a number of interesting conditions on the coefficients a_ℓ guaranteeing that the matrix $A = [G(\xi_j \cdot \xi_k)]$ will be positive definite—and therefore invertible—for *any* finite set $\{\xi_1, \ldots, \xi_N\} \subset \mathbb{S}^m$ in which the points are distinct. One of them is that all a_ℓ be positive. This condition was shown by Narcowich [17, Theorem 4.2] to be necessary *and* sufficient for positive definiteness of matrices arising in generalized Hermite interpolation, and it plays an important role here, too; it is needed in §III for density results essential to our construction. A positive definite spherical function with all a_ℓ positive provides a vehicle for doing an *intrinsic* form of interpolation on $\mathbb{S}^m$, with the interpolants being of the form

$$f(p) = \sum_{k=1}^{N} f_k G(p \cdot \xi_k), \quad f_k \in \mathbb{C}. \tag{2.2}$$

In many respects, they are the spherical analogue of order 0 radial basis functions (RBFs), whose Fourier transforms are also positive. We will call any positive definite function on $\mathbb{S}^m$ a *spherical basis function* (SBF) for $\mathbb{S}^m$ if $a_\ell > 0$ for all ℓ.

Our first result connects the positive definiteness for certain G on $\mathbb{S}^m$ with G being an SBF on $\mathbb{S}^{m'}$ with $m' < m$.

PROPOSITION 2.1: *If G is a positive definite function on $\mathbb{S}^m$, and if in the expansion (2.1) there are infinitely many even and odd ℓ's with $a_\ell > 0$, then G is an SBF on all $\mathbb{S}^{m'}$ with $m' < m$.*

PROOF: We will borrow a technique used by Ron and Sun to prove a similar result [23, Corollary 3.4]. The $m+1$-dimensional Legendre polynomial $P_\ell(m+1;t)$ can be expressed in terms of the m-dimensional Legendre polynomials via

$$P_\ell(m+1;t) = \sum_{k=0}^{\ell} a_k^{m,\ell} P_k(m;t).$$

The Legendre polynomials are even for ℓ even, and odd for ℓ odd. Consequently, when ℓ and k are opposite in parity, all $a_{2k-1}^{m,\ell} = 0$. On the other hand, when the parity of k and ℓ agree, the coefficient $a_k^{m,\ell}$ is positive. (See [24, Footnote, p. 97 and 1, p. 192]; these discuss the Gegenbauer polynomials, but the results hold here because the $m+1$ dimensional Legendre polynomials are simple multiples of the corresponding Gegenbauer polynomials.) Thus, if in (2.1) $a_\ell > 0$, then in the corresponding expansion with $m \to m-1$ all coefficients with index less than or equal to ℓ and with the same parity will be positive. Finally, in the expansion (2.1) there are infinitely many even and odd ℓ's with $a_\ell > 0$, then then for the $m \to m-1$ case all the coefficients will be positive. Hence, G is an SBF. That the result applies to all $m' < m$ is obvious. ∎

Any positive definite function on $\mathbb{S}^m$ may be expressed in a series of spherical harmonics, the standard orthonormal basis for $\mathbb{S}^m$ in the usual inner product on $L^2(\mathbb{S}^m)$. To do this, note that the famous addition theorem for spherical harmonics [8, pp. 130-133; 12, p. 172; 16, Theorem 2] takes the form,

$$P_\ell(m+1; p \cdot q) = \frac{\omega_m}{d_{m,\ell}} \sum_{j=1}^{d_{m,\ell}} Y_j^\ell(p) \bar{Y}_j^\ell(q), \tag{2.3}$$

where ω_m is the volume of $\mathbb{S}^m$ and $d_{m,\ell}$ is the dimension of the space of order-ℓ spherical harmonics on $\mathbb{S}^m$; these are given by

$$\omega_m = \frac{(2\pi)^{\frac{m+1}{2}}}{\Gamma(\frac{m+1}{2})} \quad \text{and} \quad d_{m,\ell} = \begin{cases} \dfrac{(2\ell+m-1)(\ell+m-2)!}{\ell!\,(m-1)!} & \text{if } \ell \geq 1, \\ 1 & \text{if } \ell = 0. \end{cases} \tag{2.4}$$

Using this theorem allows us to rewrite (2.1) as

$$G(p\cdot q) = \sum_{\ell=0}^{\infty} a_\ell \frac{\omega_m}{d_{m,\ell}} \sum_{j=1}^{d_{m,\ell}} Y_j^\ell(p)\bar{Y}_j^\ell(q). \tag{2.5}$$

We will make use of this formula in several of our results, the first one being to compute the integral in the following lemma.

Lemma 2.2: *If G and H have expansions of the form (2.1), with the coefficients for H corresponding to a_ℓ being denoted by b_ℓ, then*

$$\int_{\mathbb{S}^m} G(p\cdot\eta)H(\eta\cdot q)d\omega(\eta) = \sum_{\ell=0}^{\infty} a_\ell b_\ell \frac{\omega_m}{d_{m,\ell}} P_\ell(m+1; p\cdot q), \tag{2.6}$$

where $d\omega$ is the standard measure on $\mathbb{S}^m$.

Proof: Expand G and H in terms of the spherical harmonics, and use Parseval's relation and the addition theorem to get (2.6). ∎

We remark that we have not assumed that either G or H is a positive definite function, so the a_ℓ's and b_ℓ's are not necessarily nonnegative. The integral product in (2.6) will occur frequently enough here that we introduce notation for it; define the "star" product of G and H via

$$G\star H(p\cdot q) := \int_{\mathbb{S}^m} G(p\cdot\eta)H(\eta\cdot q)d\omega(\eta). \tag{2.7}$$

The star-product is computed relative to $\mathbb{S}^m$. It is easy to show that one obtains a different result by changing m. Occasionally we need to indicate the particular sphere used for the star product. When this is necessary, we will use

$$G\overset{m}{\star}H(p\cdot q) \tag{2.8}$$

instead of $G\star H$. Also, with the notation of the lemma, we have have this expansion for $G\star H$,

$$G\star H(p\cdot q) = \sum_{\ell=0}^{\infty} a_\ell b_\ell \frac{\omega_m}{d_{m,\ell}} P_\ell(m+1; p\cdot q). \tag{2.9}$$

Our next result deals with $G \star H$ when G and H are positive definite, or are SBFs.

PROPOSITION 2.3: *If G and H have expansions of the form (2.1), then $G \star H = H \star G$. Moreover, if G and H are positive definite on $\mathbb{S}^m$, then so is $G \star H$. Finally, if G and H are SBFs, then $G \star H$ is an SBF.*

PROOF: That $G \star H = H \star G$ follows from (2.9). To prove the rest of the theorem, note that $\sum_{\ell=0}^{\infty} a_\ell b_\ell \frac{\omega_m}{d_{m,\ell}} < \infty$, and that the coefficients multiplying the Legendre polynomials are nonnegative. By Schoenberg's representation theorem, $G \star H$ in (2.6) is thus a positive definite function on $\mathbb{S}^m$. Moreover, when G and H are SBFs, the a_ℓ's and the b_ℓ's are all positive, and so all of the products $a_\ell b_\ell$ are positive; $G \star H$ is thus an SBF. ∎

We have the following corollary to Proposition 2.3.

COROLLARY 2.4: *Let $\mathcal{S} = \{\xi_1, \ldots, \xi_N\}$ be N distinct points on the unit sphere, let G be an SBF, and let $\mathcal{V} = \mathrm{Span}\left\{\{G(p \cdot \xi_k)\}_{k=1}^N\right\}$. If f and g are in $\mathcal{V}$, that is, if*

$$f(p) = \sum_{k=1}^{N} f_k G(p \cdot \xi_k) \quad \text{and} \quad g(p) = \sum_{k=1}^{N} g_k G(p \cdot \xi_k), \tag{2.10}$$

then

$$\langle f, g \rangle := \int_{\mathbb{S}^m} f(\eta)\overline{g(\eta)} d\omega(\eta) = \mathbf{g}^* A_\star \mathbf{f}, \tag{2.11}$$

where $\mathbf{f}$ and $\mathbf{g}$ are column vectors with f_k's and g_k's for entries, and $A_\star$ is the interpolation matrix corresponding to the SBF $G \star G$. In addition, $A_\star$ is positive definite and $\dim \mathcal{V} = N$.

PROOF:

$$\begin{aligned}
\langle f, g \rangle &= \int_{\mathbb{S}^m} f(p)\overline{g(p)} d\omega(p) \\
&= \sum_{j,k} f_k \bar{g}_j \int_{\mathbb{S}^m} G(\xi_j \cdot \eta) G(\xi_k \cdot \eta) d\omega(\eta) \\
&= \sum_{j,k} f_k \bar{g}_j G \star G(\xi_j \cdot \xi_k) \\
&= \mathbf{g}^* A_\star \mathbf{f}
\end{aligned}$$

The matrix $A_\star$ is positive definite because $G \star G$ is an SBF, by Proposition 2.3 That $\dim \mathcal{V} = N$ is an easy consequence of $A_\star$ being positive definite. ∎

The space $\mathcal{V}$ described above is a typical sampling space. We will discuss such spaces in the next section. In the remainder of this section, we will look at three examples of SBFs. For the first two, we will explicitly compute $G \star G$. In the last example, we construct an SBF that has "local" support.

EXAMPLE 2.5: The exponential function

$$G_\lambda^{exp}(p \cdot q) = e^{\lambda p \cdot q} \tag{2.12}$$

gives rise to positive definite interpolation matrices, even when the data being interpolated comprises evaluations of distributions [17, §4]. It is thus an SBF. We want to compute $G_\lambda^{exp} \star G_\lambda^{exp}(p\cdot q)$. From (2.9), we have

$$G_\lambda^{exp} \star G_\lambda^{exp}(p\cdot q) = \int_{\mathbb{S}^m} e^{\lambda(p+q)\cdot\eta} d\omega(\eta).$$

This integral is an entire function of λ. If we let $z = i\lambda$, then the integral we have to evaluate is

$$\int_{\mathbb{S}^m} \exp\left(i \underbrace{(-z(p+q))}_{x} \cdot \eta \right) d\omega(\eta).$$

Assume that z is real and positive. The integral above is a multiple of the one found in equation (6.1) in [29, p. 26]. From equation (6.10) in [29, p. 27], with $x = -z(p+q)$, we have that

$$\int_{\mathbb{S}^m} \exp\left(-iz(p+q)\cdot\eta\right) d\omega(\eta) = \omega_m \Gamma((m+1)/2)(2/z|p+q|)^{(m-1)/2} J_{(m-1)/2}(z|p+q|).$$

where Γ is the Gamma function and J_p is the Bessel function of the first kind of order p. We want to make the change of variable $z = i\lambda$. Instead of employing J_p with an imaginary argument, we will use modified Bessel functions of the first kind, $I_p(\zeta)$. From [28, p. 77], $I_p(\zeta) = e^{-ip\pi/2} J_p(\zeta e^{i\pi/2})$, provided $-\pi < \arg\zeta \le \pi/2$. Doing so yields

$$\begin{aligned}\int_{\mathbb{S}^m} \exp\left(\lambda(p+q)\cdot\eta\right) d\omega(\eta) =& \omega_m \Gamma((m+1)/2)(2/z|p+q|)^{(m-1)/2} J_{(m-1)/2}(z|p+q|) \\ =& \omega_m \Gamma((m+1)/2)(2/\lambda|p+q|)^{(m-1)/2} I_{(m-1)/2}(\lambda|p+q|)\end{aligned}$$

Using the definition of $\star$ and the identity $|p+q| = \sqrt{2+2p\cdot q}$, from the last equation we have that

$$G_\lambda^{exp} \star G_\lambda^{exp}(p\cdot q) = \omega_m \Gamma((m+1)/2) \left(\frac{2}{\lambda\sqrt{2+2p\cdot q}} \right)^{(m-1)/2} I_{(m-1)/2}(\lambda\sqrt{2+2p\cdot q}). \tag{2.13}$$

We want to specialize this formula to the $m = 2$ case. Since $I_{1/2}(\zeta) = (2\pi\zeta)^{-1/2}(e^\zeta - e^{-\zeta}) = (2/\pi\zeta)^{1/2}\sinh(\zeta)$ (cf. [28, eq. (10), p. 80], we have, after simplifying (2.13) with $m = 2$,

$$G_\lambda^{exp} \star G_\lambda^{exp}(p\cdot q) = 4\pi \frac{\sinh(\lambda\sqrt{2+2p\cdot q})}{\lambda\sqrt{2+2p\cdot q}} \qquad (m = 2 \text{ case}). \tag{2.14}$$

Example 2.6: Generating functions for the Legendre polynomials provide a fruitful source of SBFs. From [16, Lemma 17, p. 30], we have that, for $0 < h < 1$,

$$G_h^{gen}(p\cdot q) := \frac{1-h^2}{(1+h^2-2h\,p\cdot q)^{(m+1)/2}} = \sum_{\ell=0}^{\infty} d_{m,\ell} h^\ell P_\ell(m+1; p\cdot q) \tag{2.15}$$

We can compute the $\star$ product of G_h^{gen} with itself via the series (2.9); the result is

$$\begin{aligned} G_h^{gen} \star G_h^{gen}(p\cdot q) =& \sum_{\ell=0}^{\infty} \left(d_{m,\ell} h^{\ell}\right)^2 \frac{\omega_m}{d_{m,\ell}} P_\ell(m+1; p\cdot q) \\ =& \omega_m \sum_{\ell=0}^{\infty} d_{m,\ell} \left(h^2\right)^{\ell} P_\ell(m+1; p\cdot q) \end{aligned}$$

From this series and (2.15) we obtain

$$G_h^{gen} \star G_h^{gen}(p\cdot q) = \omega_m G_{h^2}^{gen}(p\cdot q). \tag{2.16}$$

EXAMPLE 2.7: For our last example, let $0 < \alpha < \pi$. We will construct for $\mathbb{S}^m$ an SBF $G_\alpha^{loc}(p\cdot q)$ such that $G_\alpha^{loc}(p\cdot q) = 0$ when $\mathrm{Arccos}(p\cdot q) \geq \alpha$. When an SBF has this property, we shall say that it is *locally supported.* First, choose a continuous auxiliary function $B^\alpha(t)$ such that $B^\alpha(t) = 0$ when $-1 \leq t < \cos\alpha$, and such that $B^\alpha(1) \neq 0$. We wish to expand this function in a series of $m+2$ dimensional Legendre polynomials,

$$B^\alpha(t) = \sum_{\ell=0}^{\infty} b_\ell^\alpha P_\ell(m+2; t). \tag{2.18}$$

Since $B^\alpha(t)$ is not itself a polynomial, infinitely many coefficients in (2.18) will be nonzero. Next, use (2.18) to compute $B^{\alpha/2} \star B^{\alpha/2}$ with respect to $\mathbb{S}^{m+1}$; the result with is

$$B^{\alpha/2} \star B^{\alpha/2}(p\cdot q) = \sum_{\ell=0}^{\infty} (b_\ell^\alpha)^2 \frac{\omega_{m+1}}{d_{m+1,\ell}} P_\ell(m+2; p\cdot q). \tag{2.19}$$

Note that from the integral, form of the star product, $B^{\alpha/2} \star B^{\alpha/2}(p\cdot q) = 0$ if the angle between p and q on $\mathbb{S}^{m+1}$ is larger than α, for the supports of $B^{\alpha/2}(p\cdot\eta)$ and $B^{\alpha/2}(p\cdot\eta)$ are disjoint in that case. Thus, $B^{\alpha/2} \star B^{\alpha/2}(p\cdot q)$ is *locally* supported on $\mathbb{S}^{m+1}$. From the nonnegativity of the coefficients in (2.19), it also follows that it is positive definite on $\mathbb{S}^{m+1}$, with infinitely many nonvanishing coefficients. From [23, Corrolary 3.4], it is thus *strictly* positive definite—*i.e.,* an SBF—for $\mathbb{S}^m$. Consequently,

$$G_\alpha^{loc}(p\cdot q) := \int_{\mathbb{S}^{m+1}} B^{\alpha/2}(p\cdot\eta) B^{\alpha/2}(q\cdot\eta)\, d\omega(\eta) \tag{2.20}$$

is a locally supported SBF relative to $\mathbb{S}^m$.

III. Sampling Spaces and Wavelet Spaces

In this section, our aim is to construct a nonstationary multiresolution analysis for $\mathbb{S}^m$ that is based on the typical sampling space defined in Corollary 2.4. In defining that sampling space, we needed an SBF $G(p\cdot q)$, which we assume to be fixed throughout the discussion, and a set of centers. Consequently, for the sequence of sampling spaces required in the multiresolution analysis, we require a set of centers for each sampling space. Indeed, suppose that we have chosen a sequence of subsets of $\mathbb{S}^m$, $\{\mathcal{S}_k\}_{k=1}^{\infty}$, with the following properties:

(i) The cardinality of $\mathcal{S}_k \subset \mathbb{S}^m$ is N_k, and $N_k < \infty$.
(ii) Each $\mathcal{S}_k = \{\xi_1^k, \ldots \xi_{N_k}^k\}$ comprises distinct points.
(iii) The sequence is nested, $\mathcal{S}_k \subset \mathcal{S}_{k+1}$ for all $k \geq 1$. .
(iv) The union of all sets is dense in the m-sphere, $\overline{\cup_{k=1}^{\infty} \mathcal{S}_k} = \mathbb{S}^m$.

The sampling space associated with G and each $\mathcal{S}_k$ is

$$\mathcal{V}_k := \operatorname{Span}\left\{G(p \cdot \xi_j^k\right\}_{j=1}^{N_k}. \tag{3.1}$$

We summarize the properties of $\mathcal{V}_k$ below.

THEOREM 3.1: *If $\mathcal{V}_k$ is defined by (3.1), then the following hold:*

$$\dim \mathcal{V}_k = N_k \tag{3.2}$$

$$\mathcal{V}_k \subset \mathcal{V}_{k+1}, \ k = 1, 2, \ldots \tag{3.3}$$

$$\overline{\cup_{k=1}^{\infty} \mathcal{V}_k} = L^2(\mathbb{S}^m) \tag{3.4}$$

PROOF: That $\dim \mathcal{V}_k = N_k$ is a consequence of the assumptions on the centers and Corollary 2.4. Similarly, (3.3) follows directly from (iii) and the definition of $\mathcal{V}_k$. The last statement, namely that the union of the $\mathcal{V}_k$'s is dense in $L^2(\mathbb{S}^m)$, is essentially proved in [25, Theorem 2]. For $\mathbb{S}^2$, density can also be established by directly estimating a rate of convergence, as is done in [10, Corollary 3.4]. Here, we will give a brief, self-contained proof that was inspired by a proof of A. Brown (see the appendix to [21]). To prove density, we need only prove that if h is orthogonal to all of the $\mathcal{V}_k$'s, then $h \equiv 0$. Thus, we begin by supposing that if $f \in \mathcal{V}_k$, then

$$< f(\eta), h(\eta) >= 0.$$

Of course, this can happen for every $f \in \mathcal{V}_k$ if and only if $h(\eta)$ is orthogonal to $G(\eta \cdot \xi_j^k)$ for every center, ξ_j^k. Since, by (iv), the collection of all the centers forms a dense set on points on $\mathbb{S}^m$, it is clear that

$$< G(\eta \cdot \xi), h(\eta) >\equiv 0$$

for every $\xi \in \mathbb{S}^m$. Expand h using spherical harmonics,

$$h(\eta) = \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{m,\ell}} h_{\ell,j} Y_j^{\ell}(\eta).$$

Use the resulting series for h, the series for G in (2.5), and Parseval's Theorem to obtain

$$< G(\eta \cdot \xi), h(\eta) >= \sum_{\ell=0}^{\infty} \sum_{j=1}^{d_{m,\ell}} \bar{h}_{\ell,j} a_\ell \frac{\omega_m}{d_{m,\ell}} Y_j^{\ell}(\xi)$$

The coefficients in the expansion on the right are products of ℓ_2 sequences, and are therefore in ℓ_1. However, since $\ell_1 \subset \ell_2$, the series on the right defines a function $g(\xi)$ that is in $L^2(\mathbb{S}^m)$. On the other hand, $g \equiv 0$, since h is orthogonal to all $G(\eta \cdot \xi)$ for all $\xi \in \mathbb{S}^m$. Thus the coefficients in the series for g must all vanish; that is,

$$\bar{h}_{\ell,j} a_\ell \frac{\omega_m}{d_{m,\ell}} = 0$$

for all ℓ and j. Since $a_\ell > 0$ for all ℓ, we have $\bar{h}_{\ell,j} = 0$, from which it follows that $h \equiv 0$ and that $\cup_{k=1}^{\infty} \mathcal{V}_k$ is dense in $L^2(\mathbb{S}^m)$. ∎

REMARK 3.2: From the argument above, it follows that if even one $a_\ell = 0$, then the nonzero function $h = Y_{\ell,j}$ is orthogonal to $\cup_{k=1}^{\infty} \mathcal{V}_k$—meaning this space is not dense in $L^2(\mathbb{S}^m)$. Thus the union of all the $\mathcal{V}_k$'s is dense if and only if the positive definite spherical function G has all $a_\ell > 0$.

The wavelet spaces are defined as the orthogonal complement of $\mathcal{V}_k$ in $\mathcal{V}_{k+1}$; that is,

$$\mathcal{W}_k := \mathcal{V}_{k+1} \ominus \mathcal{V}_k \,. \tag{3.5}$$

Since we will be concerned only with the relationship between two adjacent levels, and among the three spaces $\mathcal{V}_k$, $\mathcal{V}_{k+1}$, and $\mathcal{W}_k$, we can simplify the notation for the various spaces and centers involved:

$$\begin{cases} \mathcal{V}_\flat := \mathcal{V}_k = \mathrm{Span}\{G(p\cdot\xi_1), \ldots, G(p\cdot\xi_N)\} \\ \mathcal{V} := \mathcal{V}_{k+1} = \mathrm{Span}\{G(p\cdot\xi_1), \ldots, G(p\cdot\xi_{N+M})\} \\ \mathcal{W} := \mathcal{W}_k \end{cases} \tag{3.6}$$

The dimensions of the three spaces are, respectively, N, $N+M$, and M. For the two sampling spaces, the “natural” bases are displayed above. The coordinate vector of $f \in \mathcal{V}$ relative to the natural basis for $\mathcal{V}$ will be denoted by $\mathbf{f}$, and for $f \in \mathcal{V}_\flat$ by $\mathbf{f}_\flat$. Since $\mathcal{V}_\flat \subset \mathcal{V}$, every $f \in \mathcal{V}_\flat$ can be represented by $\mathbf{f}_\flat$ or $\mathbf{f}$, the connection being as follows. If

$$\mathbf{f}_\flat = (\, f_1 \quad f_2 \quad \cdots \quad f_N \,)^T \in \mathbb{C}^N,$$

then

$$\mathbf{f} = (\, f_1 \quad f_2 \quad \cdots \quad f_N \quad 0 \quad \cdots \quad 0 \,)^T = (\, \mathbf{f}_\flat \quad \mathbf{0} \,)^T \in \mathbb{C}^{N+M}. \tag{3.7}$$

Next, we want to characterize those $\mathbf{g} \in \mathbb{C}^{N+M}$ that are coordinate vectors for $g \in \mathcal{W}$. Let $\mathbf{W}$ be the subspace of $\mathbb{C}^{N+M}$ comprising all such vectors. We have this result.

PROPOSITON 3.3: *Let G be an SBF for $\mathbb{S}^m$, and let $A_\star$ be the interpolation matrix associated with the $G\star G$ and the centers $\{\xi_1, \ldots, \xi_{N+M}\}$. If $\mathbf{W}$ is the subspace of $\mathbb{C}^{N+M}$ comprising all coordinate vectors for $g \in \mathcal{W}$, then the last M columns of $A_\star^{-1}$ constitute* *a basis for* $\mathbf{W}$*. In addition, if we use the* $L^2(\mathbb{S}^m)$ *inner product on* $\mathcal{W}$*, then* $\mathbf{W} \cong \mathcal{W}$*, provided one endows* $\mathbf{W}$ *with the inner product*

$$< \mathbf{g}, \mathbf{f} >_w := \mathbf{f}^* A_\star \mathbf{g}. \tag{3.8}$$

PROOF: Let $g \in \mathcal{V}$. Since $\mathcal{V} = \mathcal{V}_\flat \oplus^\perp \mathcal{W}$, g is in $\mathcal{W}$ if and only if $< g, f >= 0$ for all $f \in \mathcal{V}_\flat$. By Corollary 2.4 (with $N \to N + M$) and (3.7), we have that

$$\begin{aligned} < g, f > &= \mathbf{f}^* A_\star \mathbf{g} \\ &= (\, \mathbf{f}_\flat^* \quad \mathbf{0} \,)\, A_\star \mathbf{g}. \end{aligned}$$

The last equation can happen if and only if $\mathbf{g}$ has the form

$$\mathbf{g} = A_\star^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}, \tag{3.9}$$

where $\mathbf{w} \in \mathbb{C}^M$ is arbitrary. Standard matrix theory and the invertibilty of $A_\star^{-1}$ then imply that the last M columns of $A_\star^{-1}$ form a basis for $\mathbf{W}$. That $\mathbf{W}$ and $\mathcal{W}$ are isometrically isomorphic follows from the remarks above. ∎

REMARK 3.4: If the labeling of centers differs from that used above, then $\mathbf{W}$ is precisely the span of the columns of $A_\star^{-1}$ that correspond to the centers used in defining $\mathcal{V}$ but not in defining $\mathcal{V}_\flat$.

Suppose that h belongs to the sampling space $\mathcal{V}$, and that we want to decompose it into $f + g$, where f is in the lower level sampling space $\mathcal{V}_\flat$, and g is in the wavelet space $\mathcal{W}$. The $\mathcal{V}$-coordinates of h then break up this way:

$$\mathbf{h} = \begin{pmatrix} \mathbf{h}' \\ \mathbf{h}'' \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\flat \\ \mathbf{0} \end{pmatrix} + \underbrace{\begin{pmatrix} \mathbf{g}' \\ \mathbf{g}'' \end{pmatrix}}_{\mathbf{g}}$$

of course, from (3.9), it follows that

$$A_\star \mathbf{g} = \begin{pmatrix} \mathbf{0} \\ \mathbf{w} \end{pmatrix}. \tag{3.10}$$

Write $A_\star$ in partitioned form,

$$A_\star = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

where A is an $N \times N$ matrix (actually, "$A_\star^\flat$"), C is an $M \times M$, and B is an $N \times M$. Because $A_\star$ is positive definite, the matrices A and C are too. Multiplying the partitioned form of $A_\star$ times the partitioned form of $\mathbf{g}$, and using (3.10), we find that $A\mathbf{g}' + B\mathbf{g}'' = \mathbf{0}$. Consequently,

$$\mathbf{g}' = -A^{-1} B \mathbf{g}''. \tag{3.11}$$

Combining these formulas gives us a way of obtaining $\mathbf{f}_\flat$ and $\mathbf{g}$ from $\mathbf{h}$ and visa versa—essentially *decomposition* and *reconstruction* formulas—, which we state below.

PROPOSITION 3.5: *With the notation introduced above, we have*

$$\left\{\begin{aligned} \mathbf{g}'' &= \mathbf{h}'', \\ \mathbf{g}' &= -A^{-1}B\mathbf{h}'', \\ \mathbf{f}_\flat &= \mathbf{h}' + A^{-1}B\mathbf{h}''. \end{aligned}\right. \qquad \mathbf{h} = \begin{pmatrix} \mathbf{h}' \\ \mathbf{h}'' \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\flat \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{g}' \\ \mathbf{g}'' \end{pmatrix} \tag{3.12}$$

PROOF: The three bracketed equations constitute the decomposition formula. The first of the three follows from (3.9), and the second from (3.11). The last follows from (3.9) and the first and second equations. The reconstruction formula was actually mentioned earlier; it is obtained by adding the contribution from $\mathcal{V}_\flat$ to that from $\mathcal{W}$. ∎

We now turn to a discussion of "spherical wavelets," by which we mean vectors in $\mathcal{W}$ that are in some sense localized and form a basis for $\mathcal{W}$. If possible, one would like the basis to be orthonormal as well. In the next section, we discuss localization criteria. Here, we will give ways of constructing all bases for $\mathcal{W}$, including all orthonormal bases.

Recall from Proposition 3.3 that the last M columns of $A_\star^{-1}$ constitute a basis for $\mathbf{W}$. Let us partition $A_\star^{-1}$,

$$A_\star^{-1} = \begin{pmatrix} R & S \\ S^* & Q \end{pmatrix}, \tag{3.13}$$

where R is $N \times N$, S is $N \times M$, and Q is $M \times M$. Since $A_\star$ is positive definite, both R and Q are positive definite. Also, from (3.13), we have

$$A_\star \underbrace{\begin{pmatrix} S \\ Q \end{pmatrix}}_{U} = \begin{pmatrix} 0_{N\times M} \\ I_{M\times M} \end{pmatrix}. \tag{3.14}$$

Of course, the $(N+M)\times M$ matrix U comprises the last M columns of $A_\star^{-1}$. From standard matrix theory, we have that another $(N+M)\times M$ matrix Υ will have columns forming a basis for $\mathbf{W}$ if and only if there is an invertible $M \times M$ matrix P such that

$$\Upsilon = UP. \tag{3.15}$$

If we demand not only that the columns of Υ form a basis, but an *orthonormal* basis, then the choice of P becomes greatly restricted.

PROPOSITION 3.6: *Let U be the $(N+M)\times M$ matrix comprising the last M columns of $A_\star^{-1}$, and let Q be the $M\times M$ "southeast" block of $A_\star^{-1}$. An invertible matrix P is such that the columns of $\Upsilon = UP$ comprise an orthonormal basis for $\mathbf{W}$ relative to $<\cdot,\cdot>_w$ if and only if*

$$PP^* = Q^{-1} \tag{3.16}$$

PROOF: From the matrix form of the inner product $< \cdot, \cdot >_w$ given in (3.8), we see that requiring the columns of Υ to be orthonormal is equivalent to the matrix equation

$$\Upsilon^* A_\star \Upsilon = I_{M\times M}, \tag{3.17}$$

where $I_{M\times M}$ is the $M \times M$ identity matrix. From $\Upsilon = UP$ and (3.17), we obtain

$$P^* U^* A_\star UP = I_{M\times M}$$

$$P^* \underbrace{U^*}_{(\,S^* \quad Q\,)} \begin{pmatrix} 0_{N\times M} \\ I_{M\times M} \end{pmatrix} P = I_{M\times M} \quad \text{(from (3.14))}$$

$$P^* Q P = I_{M\times M}$$

Equation (3.16) is a trivial consequence of the last equation. ∎

One of the reviewers posed the following interesting question. The utility of wavelets in euclidean space stems in part from being able to characterize function spaces by means of wavelet coefficients. This characterization is made possible by the existence of dilation operators in euclidean space. Of course, no such operators exist *globally* on the m-sphere. However, they do exist *locally* in coordinate patches. Do the wavelets constructed above reduce to standard euclidean wavelets in sufficiently small coordinate patches? To what extent does one recover the euclidean results in such patches?

Holschneider [13] has addressed the question of the euclidean limit and obtained interesting results for the case of the continuous wavelet transform. For the wavelets discussed here, we do not have a precise answer to these questions. However, we can say a few words about it. The SBFs in the small behave very much like radial basis functions. For example, $e^{\lambda\cos\theta} \approx e^{\lambda}e^{\lambda\theta^2/2}$ when the arclength θ is small. The SBF $e^{\lambda}\cos\theta$ behaves like a Gaussian in this case. We therefore conjecture that the wavelets constructed out of $e^{\lambda\cos\theta}$ will asymptotically behave like the corresponding RBF wavelets constructed from Gaussians. Whether one can obtain an SBF that yields a Daubechies-like wavelet is an important and interesting open question.

IV. Localization

We will now investigate how well the wavelets constructed out of SBFs localize on the sphere $\mathbb{S}^m$. The chief measure of localization for a function f defined on the line is the variance (or its square-root: standard deviation),

$$\mathrm{var}_f(t) = \int_{\mathbb{R}} (t-t_0)^2 |f(t)|^2 dt, \quad \text{where} \quad t_0 := \int_{\mathbb{R}} t|f(t)|^2 dt.$$

The notion of variance for functions defined on $\mathbb{S}^m$ is more subtle than variance on the line or in $\mathbb{R}^s$, simply because one cannot smoothly cover $\mathbb{S}^m$ with a single set of coordinates. In addition, it must be invariant under rotations. Breitenberger [2] discusses both a notion of variance and an uncertainty principle for the circle $\mathbb{S}^1$. We have adapted these to deal with localization and uncertainty for periodic wavelets discussed in [18]. The definition and interpretation of variance used in [2] and [18] can easily be extended to $\mathbb{S}^m$. The uncertainty principle is another matter; we will discuss it in §V.

In what follows, let f be in $L^2(\mathbb{S}^m)$, and let $\|f\|_{L^2} = 1$; set

$$\mathfrak{r}_f := \int_{\mathbb{S}^m} \eta |f(\eta)|^2 d\omega(\eta) \tag{4.1}$$

We remark that we are regarding $\mathbb{S}^m$ as embedded in $\mathbb{R}^{m+1}$, and η as both the normal to the sphere and a point on the sphere. We have *not* chosen coordinates on $\mathbb{S}^m$. One may interpret $|f(\eta)|^2 d\omega(\eta)$ as a mass distribution in $\mathbb{R}^{m+1}$ that is confined to $\mathbb{S}^m$, with $\mathfrak{r}_f$ being the center of mass of the distribution. The corresponding variance is easily seen to be

$$\mathrm{var}_f(\eta) = \int_{\mathbb{S}^m} \|\eta - \mathfrak{r}_f\|^2 |f(\eta)|^2 d\omega(\eta), \tag{4.2}$$

which is defined on $\mathbb{S}^m$ and is invariant both choice of coordinates and rotations. Since

$$\|\eta - \mathfrak{r}_f\|^2 = \eta \cdot \eta - 2\eta \cdot \mathfrak{r}_f + \mathfrak{r}_f \cdot \mathfrak{r}_f = 1 + \|\mathfrak{r}_f\|^2 - 2\eta \cdot \mathfrak{r}_f,$$

we may integrate both sides to get

$$\mathrm{var}_f(\eta) = 1 - \|\mathfrak{r}_f\|^2 \tag{4.3}$$

The following inequalities are immediate consequences of (4.1)-(4.3):

$$0 \le \|\mathfrak{r}_f\| \le 1 \quad \text{and} \quad 0 \le \mathrm{var}_f(\eta) \le 1. \tag{4.4}$$

We now turn to finding expressions for $\mathfrak{r}_f$ when f belongs to the sampling spaces defined earlier. To simplify matters, we will again adopt the conventions used throughout the last half of §III, starting with those in (3.6). Our first result is the following.

Proposition 4.1: *Let $f \in \mathcal{V}$ and let $\mathbf{f} = (\, f_1 \quad \cdots \quad f_{N+M}\,)^T$ be the coordinate vector for f relative to the basis $\{G(p \cdot \xi_j)\}_{j=1}^{N+M}$. Then, we have*

$$\mathfrak{r}_f = \sum_{j,k=1}^{N+M} \bar{f}_j f_k \int_{\mathbb{S}^m} \eta G(\eta \cdot \xi_j) G(\eta \cdot \xi_k) d\omega(\eta), \tag{4.5}$$

Proof: Express f in terms of the basis $\{G(p \cdot \xi_j)\}_{j=1}^{N+M}$, substitute the result into (4.1), and collect terms. The final result is (4.5). ∎

The quantity $\mathfrak{r}_f$ is of great importance to our theory of localization. In view of (4.5), to compute it requires simplifying vector-valued integrals of the form

$$\mathfrak{r}(p, q) := \int_{\mathbb{S}^m} \eta G(\eta \cdot p) G(\eta \cdot q) d\omega(\eta), \tag{4.6}$$

where p and q are unit vectors. We carry this out in the proposition below.

PROPOSITION 4.2: *Let G be an SBF. The vector-valued function $\mathfrak{r}(p,q)$ defined in (4.6) has the structure*

$$
\mathfrak{r}(p,q) = \begin{cases} \dfrac{T(p\cdot q)}{1+p\cdot q}\,(p+q), & p\cdot q \neq -1 \\ 0 & p\cdot q = -1, \end{cases} \tag{4.7}
$$

where T is a scalar-valued function having the form

$$
T(p\cdot q) = H \star G(p\cdot q), \text{ with } H(p\cdot q) := (p\cdot q)G(p\cdot q). \tag{4.8}
$$

PROOF: Let s be any unit vector in $\mathbb{R}^{m+1}$ that is perpendicular to p and q. We want to show that the component of $\mathfrak{r}(p,q)$ along s is 0. Choose coordinates in $\mathbb{R}^{m+1}$ so that the x_1-axis is parallel to s, and the x_2-x_3 plane contains p and q. Introducing spherical coordinates as in [29, p. 25-26], we see that

$$
\begin{aligned}
s\cdot\mathfrak{r}(p,q) &= \int_{\mathbb{S}^m} s\cdot\eta G(\eta\cdot p)G(\eta\cdot q)d\omega(\eta) \\
&= \int_0^\pi \cos(\theta_1)h(\sin\theta_1)d\theta_1 \\
&= 0,
\end{aligned}
$$

where the last equation follows from symmetry—$\cos\theta_1$ is odd about $\pi/2$ and $h(\sin\theta_1)$ is even about $\pi/2$, making the product of the two functions odd. The integral involved is therefore 0, and, consequently, $\mathfrak{r}(p,q)$ is in the plane spanned by p and q; that is, $\mathfrak{r}(p,q) = \alpha p + \beta q$. To determine α and β, we need to compute $p\cdot\mathfrak{r}(p,q)$ and $q\cdot\mathfrak{r}(p,q)$. To begin, "dot" p into both sides of (4.6), and observe that the result is precisely a "star" product

$$
p\cdot\mathfrak{r}(p,q) = H\star G(p\cdot q), \text{ with } H(p\cdot q) := (p\cdot q)G(p\cdot q). \tag{4.9}
$$

An identical computation shows that

$$
q\cdot\mathfrak{r}(p,q) = G\star H(p\cdot q) \tag{4.10}
$$

By Proposition 2.3, $G\star H = H\star G$, so

$$
q\cdot\mathfrak{r}(p,q) = p\cdot\mathfrak{r}(p,q) = G\star H(p\cdot q) =: T(p\cdot q). \tag{4.11}
$$

As long as $p\cdot q\neq -1$, it is very easy to see that $\alpha=\beta=T/(1+p\cdot q)$. Consequently, (4.7) holds for $p\cdot q\neq -1$. If $p\cdot q = -1$—equivalently, $p=-q$, then we may use (4.11) together with the obvious symmetry, $\mathfrak{r}(p,q)=\mathfrak{r}(q,p)$, to get $T(p\cdot q)=0$. This is sufficient to imply that $\mathfrak{r}(p,-p)=0$. ∎

REMARK 4.3: It is possible to show that T in (4.8) is an SBF. The proof hinges on the fact that recurrence relation for the $P_\ell(m+1;t)$'s [12, p. 181, eq. 26],

$$
(2\ell+m-1)tP_\ell(m+1;t) = (\ell+m-1)P_{\ell+1}(m+1;t) + \ell P_{\ell-1}(m+1;t),
$$

which holds for $\ell \geq 0$, provided $P_{-1} := 0$. Using this and the properties of Legender polynomials, one has that for $H(t) = tG(t)$,

$$
\begin{aligned}
H(t) = tG(t) =& \sum_{\ell=0}^{\infty} a_\ell t P_\ell(m+1;t) \\
=& \sum_{\ell=0}^{\infty} a_\ell \left(\frac{\ell+m-1}{2\ell+m-1} P_{\ell+1}(m+1;t) + \frac{\ell}{2\ell+m-1} P_{\ell-1}(m+1;t) \right) \\
=& \frac{a_1}{m+1} P_0(m+1;t) + \sum_{\ell=1}^{\infty} \left(\frac{a_{\ell-1}(\ell+m-2)}{2\ell+m-3} + \frac{a_{\ell+1}(\ell+1)}{2\ell+m+1} \right) P_\ell(m+1;t) \qquad (4.12)
\end{aligned}
$$

Thus the coefficients in (4.1) are strictly positive, and one may also check that are summable. Thus, H is an SBF. Of course, we have assumed that G is an SBF, so by Propostion 2.3, $T = G \star H$ is also an SBF.

Combining Propositions 4.1 and 4.2 yields a formula for $\mathfrak{r}_f$ when f is in $\mathcal{V}$.

THEOREM 4.4: *With the notation of Propositions 4.1 and 4.2, we have*

$$
\mathfrak{r}_f = \sum_{j,k=1}^{N+M} \bar{f}_j f_k \frac{T(\xi_j \cdot \xi_k)}{1 + \xi_j \cdot \xi_k} (\xi_j + \xi_k), \qquad (4.13)
$$

where it is understood that any term for which $\xi_k = -\xi_j$ is 0.

PROOF: Substitute (4.7) into (4.5). ∎

EXAMPLE 4.5: When $G = G_\lambda^{exp}$, which is defined in (2.12), we can directly compute the function T in (4.8). To do this, first observe that since $T = G \star H = H \star G$, where $H_\lambda^{exp}(p \cdot q) = (p \cdot q) G_\lambda^{exp}(p \cdot q)$, we have the following chain of equations:

$$
\begin{aligned}
T_\lambda^{exp}(p \cdot q) &= \frac{1}{2} G_\lambda^{exp} \star H_\lambda^{exp}(p \cdot q) + \frac{1}{2} H_\lambda^{exp} \star G_\lambda^{exp}(p \cdot q) \\
&= \frac{1}{2} \int_{\mathbb{S}^m} \eta \cdot (p+q) G_\lambda^{exp}(p \cdot \eta) G_\lambda^{exp}(q \cdot \eta) d\omega(\eta) \\
&= \frac{1}{2} \int_{\mathbb{S}^m} \eta \cdot (p+q) e^{\lambda (p+q) \cdot \eta} d\omega(\eta) \\
&= \frac{1}{2} \frac{\partial}{\partial \lambda} \left(\int_{\mathbb{S}^m} e^{\lambda (p+q) \cdot \eta} d\omega(\eta) \right) \\
&= \frac{1}{2} \frac{\partial}{\partial \lambda} \left(G_\lambda^{exp} \star G_\lambda^{exp}(p \cdot q) \right) . \qquad (4.14)
\end{aligned}
$$

Equation (4.14) holds for arbitrary m. If we specialize the result to $\mathbb{S}^2$, then by differentiating (2.14) we obtain the following expression for T_λ^{exp}:

$$
T_\lambda^{exp}(p \cdot q) = 2\pi \left(\frac{\lambda \cosh(\lambda \sqrt{2 + 2p \cdot q}) - \sinh(\lambda \sqrt{2 + 2p \cdot q})}{\lambda^2 \sqrt{2 + 2p \cdot q}} \right) \quad (m = 2). \qquad (4.15)
$$

EXAMPLE 4.6: We will now look at $G = G_h^{gen}$, which is defined in (2.15). To compute the function T_h^{gen} corresponding to it, we will begin by putting $H_h^{gen}(t) = tG_h^{gen}(t)$ into a form that will be useful for computing $T_h^{gen} = H_h^{gen} \star G_h^{gen}$ via (2.9).

$$\begin{aligned}
H_h^{gen}(t) = tG_h^{gen}(t) &= \frac{1-h^2}{2h}\,\frac{2ht - 1 - h^2 + 1 + h^2}{(1+h^2-2h\,t)^{(m+1)/2}} \\
&= \frac{1+h^2}{2h}G_h^{gen}(t) - \frac{1-h^2}{2h}\,\underbrace{\frac{1}{(1+h^2-2h\,t)^{(m-1)/2}}}_{F_h^{gen}(t)} \qquad (4.16) \\
&= \frac{1+h^2}{2h}G_h^{gen}(t) - \frac{1-h^2}{2h}F_h^{gen}(t) \qquad (4.17)
\end{aligned}$$

From (4.17) and (2.16), we see that

$$T_h^{gen} = H_h^{gen} \star G_h^{gen} = \frac{1+h^2}{2h}\omega_m G_{h^2}^{gen} - \frac{1-h^2}{2h}F_h^{gen} \star G_h^{gen} \qquad (4.18)$$

The function F_h^{gen} is known to have the following expansion [16, p. 33]:

$$F_h^{gen}(t) = \sum_{\ell=0}^{\infty} \binom{\ell+m-2}{m-2} h^\ell P_\ell(m+1;t), \qquad (4.19)$$

which of course makes it an SBF. Inserting the expansions for G_h^{gen} and for F_h^{gen} in the expansion (2.9) for the star product yields

$$\begin{aligned}
F_h^{gen} \star G_h^{gen}(t) &= \sum_{\ell=0}^{\infty} \binom{\ell+m-2}{m-2} h^\ell d_{m,\ell} h^\ell \frac{\omega_m}{d_{m,\ell}} P_\ell(m+1;t) \\
&= \omega_m \left(\sum_{\ell=0}^{\infty} \binom{\ell+m-2}{m-2} h^{2\ell} P_\ell(m+1;t) \right) \\
&= \omega_m F_{h^2}^{gen} \qquad (4.20)
\end{aligned}$$

Replacing $F_h^{gen} \star G_h^{gen}$ by the right side of (4.20) we obtain

$$T_h^{gen}(t) = \omega_m \left(\frac{1+h^2}{2h} G_{h^2}^{gen} - \frac{1-h^2}{2h} F_{h^2}^{gen} \right) \qquad (4.21)$$

We now want to manipulate the right side of (4.21). First, multiply by $(1+h^2)/(1+h^2)$. The coefficient of $G_{h^2}^{gen}$ is then $(1+h^2)^2 = 1+h^4+2h^2$, and that for $F_{h^2}^{gen}$ is $1-h^4$—together with factors multiplying both functions. Collecting terms and rewriting gives us

$$\begin{aligned}
T_h^{gen}(t) &= \frac{h\omega_m}{1+h^2}\left(\frac{1+h^4}{2h^2} G_{h^2}^{gen}(t) - \frac{1-h^4}{2h^2} F_{h^2}^{gen}(t) + G_{h^2}^{gen}(t) \right) \\
&= \frac{h\omega_m}{1+h^2}\left(tG_{h^2}^{gen}(t) + G_{h^2}^{gen}(t) \right) \quad \text{[from (4.17)]} \\
&= \frac{h(1+t)\omega_m}{1+h^2} G_{h^2}^{gen}(t) \qquad (4.22) \\
&= \frac{h(1+t)}{1+h^2} G_h^{gen} \star G_h^{gen}(t) \quad \text{[from (2.16)]} \qquad (4.23)
\end{aligned}$$ ∎

The last result is quite interesting because it enables us to obtain explicit formulas for $\mathfrak{r}_f$, with $f \in \mathcal{V}$, when $G = G_h^{gen}$. In that case, we may substitute (4.23) into (4.13) to obtain

$$\mathfrak{r}_f = \sum_{j,k=1}^{N+M} \bar{f}_j f_k \frac{h}{1+h^2} G_h^{gen} \star G_h^{gen}(\xi_j \cdot \xi_k)(\xi_j + \xi_k), \tag{4.24}$$

Recall that $A_\star$, which was defined in Proposition 3.3, is the interpolation matrix corresponding to $G \star G$ and associated with the centers $\xi_1, \ldots, \xi_{N+M}$. Set $G = G_h^{gen}$ and let $A_\star^h$ denote $A_\star$ for this case. Equation (4.24) can then be rewritten as

$$\mathfrak{r}_f = \frac{2h}{1+h^2} \Re \left\{ \sum_{j=1}^{N+M} \bar{f}_j [A_\star^h \mathbf{f}]_j \xi_j \right\}. \tag{4.25}$$

With this formula in hand, we can now show that for $G = G_h^{gen}$ a wavelet formed from one of the final M columns of ${A_\star^h}^{-1}$ will localize about the center corresponding to the column. Specifically, we have following localization result.

Corollary 4.7: *Let $G = G_h^{gen}$, and let $\{g_k\}_{k=1}^M$ be the basis for $\mathcal{W}$ defined by the columns of ${A_\star^h}^{-1}$; that is, the coordinate-vector of g_k relative to $\mathcal{V}$ is $\mathbf{g}_k = c_k {A_\star^h}^{-1} \mathbf{e}_{N+k}$, with c_k being a posive number chosen so that $\|g_k\| = 1$.Then,*

$$\mathfrak{r}_{g_k} = \frac{2h}{1+h^2} \xi_{N+k} \tag{4.26}$$

and

$$\mathrm{var}_{g_k}(\eta) = \left(\frac{1-h^2}{1+h^2} \right)^2. \tag{4.27}$$

Proof: Substitute $\{g_k\}_{k=1}^M$ into (4.25), and use the fact that the j^{th} component of $\mathbf{e}_{N+k}$ is $[\mathbf{e}_{N+k}]_j = \delta_{N+k,j}$ to obtain

$$\mathfrak{r}_{g_k} = \frac{2h}{1+h^2} \Re \left\{ \sum_{j=1}^{N+M} c_k \overline{[\mathbf{g}_k]}_j \delta_{N+k,j} \xi_j \right\}.$$

Carrying out the sum over j and using the fact the components of $\mathbf{g}_k$ are real (they are entries in the inverse of a real matrix), we get

$$\mathfrak{r}_{g_k} = \frac{2h}{1+h^2} c_k [\mathbf{g}_k]_{N+k} \xi_{N+k,j} \,. \tag{4.28}$$

To finish the proof, recall that, by Propositon 3.3, $\|g_k\|^2 = \mathbf{g}_k^* A_\star^h \mathbf{g}_k$. Hence, $\|g_k\|^2 = (c_k)^2 \mathbf{e}_{N+k}^T {A_\star^h}^{-1} \mathbf{e}_{N+k}$. Thus,

$$(c_k)^2 [{A_\star^h}^{-1}]_{N+k,N+k} = 1$$

Since $[\mathbf{g}_k]_{N+k} = c_k[A_\star^{h^{-1}}]_{N+k,N+k}$, the last equation implies that

$$c_k[\mathbf{g}_k]_{N+k} = (c_k)^2[A_\star^{h^{-1}}]_{N+k,N+k} = 1.$$

Inserting this in (4.28) results in (4.26). Equation (4.27) is a consequence of (4.26), (4.3), and $\|\xi_k\| = 1$. ∎

The results in Corollary 4.7 imply that the "expected value" of η for a wavelet g_k points along the center ξ_{N+k}, and that the variance depends *only* on a single variable, h. Are these results unique to the SBF G_h^{gen}? In the sharp form given in in the corollary above, it is quite likely that they are. However, we conjecture that if an SBF G has localization properties similar to G_h^{gen}, then equations (4.26) and (4.27) will hold approximately.

V. An Uncertainty Relation

We want to derive an "uncertainty relation" for the sphere $\mathbb{S}^2$. The uncertainty relation for $\mathbb{S}^1$, the circle, measures a trade-off between angular localization and angular momentum (quantum case [2]) or between angular localization and frequency (periodic wavelet case [18]). For $\mathbb{S}^2$, there exist quantum mechanical uncertainty relations, but they involve trade-offs between components of the angular momentum operator relative to different axes [15, p. 167]. The uncertainty relation that we derive here will measure the trade-off between localization on $\mathbb{S}^2$ and a "total" angular frequency, which is related to what is termed total angular momentum [15] and which we will define precisely below.

The approach that we will take is to use the standard analysis of rotations in $\mathbb{R}^3$, and then restrict various quantities to $\mathbb{S}^2$. We begin by noting that if $f(p)$ is a smooth function on $\mathbb{S}^2$, then it is easily extended to a smooth, compactly supported function defined on $\mathbb{R}^3$ via $f_{ext}(x) := f(x/\|x\|)\rho(\|x\|)$, where $\rho : [0,\infty) \to \mathbb{R}$ is a C^∞ function that is 1 on $[1/2, 2]$ and 0 on $[0, 1/4] \cup [3, \infty)$. (The specific values 1/2, 3, and so on are not crucial.)

Infinitesimal rotations about an axis pointing along a unit vector $\mathbf{n}$ are generated by the differential operator $\mathbf{n} \cdot \Omega$, where

$$\Omega := -ix \times \nabla \tag{5.1}$$

is a vector operator obtained by formally calculating the cross product. In quantum mechanics, the angular momentum operator is $\mathbf{L} = \hbar\Omega$ [15]. If one introduces spherical coordinates in $\mathbb{R}^3$ (see [7, p. 141] or [15, p. 172]), and if one chooses the z-axis to point along $\mathbf{n}$, then $\mathbf{n}\cdot\Omega = -i\partial/\partial\varphi$, the angle φ being the longitude. In any event, it is straightforward, if tedious, to show that in spherical coordinates Ω contains no radial derivatives, and at a point x has no component along the direction $x/\|x\|$. Here is a summary of the properties of Ω; these may be found in [15, p. 172-3] or deduced from standard vector analysis.

(P1) The cartesian components of Ω, which are Ω_1, Ω_2, and Ω_3, are self-adjoint operators on $L^2(\mathbb{S}^2)$.

(P2) $\Omega \cdot \Omega := \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = -\Delta_{\mathbb{S}^2}$, where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator for $\mathbb{S}^2$. (See [15, p. 173, eq. (9.34c)].)

(P3) If $\mathfrak{r}$ and $\mathfrak{o}$ are arbitrary (constant) vectors in $\mathbb{R}^3$, then we have the following identities:

$$2ix = (x - \mathfrak{r}) \times (\Omega - \mathfrak{o}) + (\Omega - \mathfrak{o}) \times (x - \mathfrak{r}) \tag{5.2}$$

$$(\Omega - \mathfrak{o}) \times (x - \mathfrak{r}) = -\left\{(x - \mathfrak{r}) \times (\Omega - \mathfrak{o})\right\}^* \tag{5.3}$$

The identities above are all that we will need to get our uncertainty principle, which appears below.

Theorem 5.1: *If $\psi(p)$ is a twice-continuously differentiable complex-valued function on $\mathbb{S}^2$, and if*

$$\int_{\mathbb{S}^m} |\psi(\eta)|^2 d\omega(\eta) = 1,\ \ \mathfrak{r}_\psi := \int_{\mathbb{S}^2} \eta|\psi(\eta)|^2 d\omega(\eta),\ \text{and}\ \mathfrak{o}_\psi := \int_{\mathbb{S}^2} \overline{\psi(\eta)}(\Omega\psi)(\eta) d\omega(\eta) \tag{5.4}$$

then

$$(5.5) \qquad \|\mathfrak{r}_\psi\|^2 \leq \left(1 - \|\mathfrak{r}_\psi\|^2\right) \mathrm{var}_\psi(\Omega), \quad \text{where } \mathrm{var}_\psi(\Omega) := \int_{\mathbb{S}^2} \|(\Omega - \mathfrak{o}_\psi)\psi(\eta)\|^2 d\omega(\eta).$$

PROOF: Extend ψ smoothly to $\mathbb{R}^3$ as mentioned above. Apply (5.2) to ψ, with the constant vectors chosen to be $\mathfrak{r} = \mathfrak{r}_\psi$ and $\mathfrak{o} = \mathfrak{o}_\psi$. (Since the components of Ω are self-adjoint, the vector $\mathfrak{o}_\psi$ has real components.) Multiply the resulting equation by $\bar{\psi}$, and integrate over the unit sphere. Observe that the right side of (5.2) becomes $\mathfrak{r}_\psi$ after carrying out these steps. In addition, using (5.3) allows us to put the whole equation into the form

$$(5.6) \qquad 2i\mathfrak{r}_\psi = \int_{\mathbb{S}^2} (2i)\Im\left\{\overline{\psi(\eta)}(\eta - \mathfrak{r}_\psi) \times (\Omega - \mathfrak{o}_\psi)\psi(\eta)\right\} d\omega(\eta).$$

After dividing by $2i$ and moving $\overline{\psi(\eta)}$ past $\eta - \mathfrak{r}_\psi$, we obtain

$$(5.7) \qquad \mathfrak{r}_\psi = \Im\left\{\int_{\mathbb{S}^2} \left((\eta - \mathfrak{r}_\psi)\overline{\psi(\eta)}\right) \times (\Omega - \mathfrak{o}_\psi)\psi(\eta) d\omega(\eta)\right\}.$$

From (5.7) and the inequality $\|a \times b\| \leq \|a\|\|b\|$ we find that

$$\|\mathfrak{r}_\psi\| \leq \int_{\mathbb{S}^2} \|(\eta - \mathfrak{r}_\psi)\overline{\psi(\eta)}\|\|(\Omega - \mathfrak{o}_\psi)\psi(\eta)\| d\omega(\eta).$$

Squaring both sides and applying Schwarz's inequality, we arrive at

$$(5.8) \qquad \|\mathfrak{r}_\psi\|^2 \leq \left(\int_{\mathbb{S}^2} \|(\eta - \mathfrak{r}_\psi)\|^2 |\psi(\eta)|^2 d\omega(\eta)\right) \left(\int_{\mathbb{S}^2} \|(\Omega - \mathfrak{o}_\psi)\psi(\eta)\|^2 d\omega(\eta)\right).$$

From (4.2), the first factor on the left above is $\mathrm{var}_\psi(\eta) = 1 - \|\mathfrak{r}_\psi\|^2$. Using this in (5.8) yields (5.5). ∎

REMARK 5.2: If $\mathfrak{r}_\psi$ is nonvanishing, then we may rewrite the results of the last theorem in a more suggestive way. Namely, if we define

$$(5.9) \qquad \Delta\eta := \frac{\sqrt{1 - \|\mathfrak{r}_\psi\|^2}}{\|\mathfrak{r}_\psi\|} \quad \text{and} \quad \Delta\Omega =: \sqrt{\mathrm{var}_\psi(\Omega)}$$

then uncertainty principle becomes

$$(5.10) \qquad \Delta\eta\Delta\Omega \geq 1.$$

It is not known whether the value 1 is best possible.

In a quantum mechanical setting, the quantity $\hbar\mathrm{var}_\psi(\Omega)$ is the variance in the expected value of the angular momentum operator. Here, $\mathrm{var}_\psi(\Omega)$ is to be interpreted as the variance in “total angular frequency.” Our next result will be to express this quantity in a simpler, more useful way.

PROPOSITION 5.3: *With the notation of Theorem 5.1, we have*

$$\Delta\Omega^2 = \mathrm{var}_\psi(\Omega) = \int_{\mathbb{S}^2} \overline{\psi(\eta)}(-\Delta_{\mathbb{S}^2})\psi(\eta)d\omega(\eta) - \|\mathfrak{o}_\psi\|^2. \tag{5.11}$$

In addition, if ψ is a real-valued function having the expansion

$$\psi(p) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1} \psi_{\ell,k} Y_{\ell,k}(p) \tag{5.12}$$

then

$$\mathfrak{o}_\psi = \mathbf{0} \quad \text{and} \quad \Delta\Omega^2 = \mathrm{var}_\psi(\Omega) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1} \ell(\ell+1)|\psi_{\ell,k}|^2 \tag{5.13}$$

PROOF: To compute the $\mathrm{var}_\psi(\Omega)$, observe that

$$\begin{aligned}
\mathrm{var}_\psi(\Omega) &= \int_{\mathbb{S}^2} \|(\Omega - \mathfrak{o}_\psi)\psi(\eta)\|^2 d\omega(\eta) \\
&= \int_{\mathbb{S}^2} \|\Omega\psi(\eta)\|^2 d\omega(\eta) - 2\mathfrak{o}_\psi \cdot \Re\left\{\int_{\mathbb{S}^2} \overline{\psi(\eta)}\Omega\psi(\eta)d\omega 9\eta)\right\} + \|\mathfrak{o}_\psi\|^2 \\
&= \int_{\mathbb{S}^2} \|\Omega\psi(\eta)\|^2 d\omega(\eta) - 2\mathfrak{o}_\psi \cdot \mathfrak{o}_\psi + \|\mathfrak{o}_\psi\|^2 \\
&= \int_{\mathbb{S}^2} \|\Omega\psi(\eta)\|^2 d\omega(\eta) - \|\mathfrak{o}_\psi\|^2 \qquad (5.14)
\end{aligned}$$

Next, we will evaluate the integral on the right in (5.14). To that end, note that $\|\Omega\psi(\eta)\|^2 = |\Omega_1\psi(\eta)|^2 + |\Omega_2\psi(\eta)|^2 + |\Omega_3\psi(\eta)|^2$, so that

$$\int_{\mathbb{S}^2} \|\Omega\psi(\eta)\|^2 d\omega(\eta) = \sum_{j=1}^{3} \int_{\mathbb{S}^2} |\Omega_j\psi(\eta)|^2 d\omega(\eta).$$

Using the selfadjointness of the Ω_j's, we can rewrite this as

$$\int_{\mathbb{S}^2} \|\Omega\psi(\eta)\|^2 d\omega(\eta) = \int_{\mathbb{S}^2} \overline{\psi(\eta)} \left(\sum_{j=1}^{3} \Omega_j^2\right) \psi(\eta)d\omega(\eta).$$

By property P2 and the last equation, we find that

$$\int_{\mathbb{S}^2} \|\Omega\psi(\eta)\|^2 d\omega(\eta) = \int_{\mathbb{S}^2} \overline{\psi(\eta)}(-\Delta_{\mathbb{S}^2})\psi(\eta)d\omega(\eta). \tag{5.15}$$

Combining (5.14) and (5.15) yields (5.11).

Assume that $\psi(p)$ is a real-valued function, and let the polar axis be chosen so that it is along an arbitrary vector $\mathbf{n}$. With this choice of axis, recall that $\mathbf{n}\cdot\Omega = -i\partial/\partial\varphi$. Using the corresponding choice of spherical coordinates, we have

$$\begin{aligned}
\mathbf{n}\cdot\mathfrak{o}_\psi &= \int_{\mathbb{S}^2} \psi(\eta)\mathbf{n}\cdot\Omega\psi(\eta)d\omega(\eta) \\
&= -i\int_0^\pi\int_0^{2\pi} \psi(\theta,\varphi)\frac{\partial\psi}{\partial\varphi}(\theta,\varphi)\sin(\theta)d\varphi d\theta \\
&= -i\int_0^\pi\int_0^{2\pi} \frac{1}{2}\frac{\partial}{\partial\varphi}\left(\psi(\theta,\varphi)^2\right)\sin(\theta)d\varphi d\theta \\
&= -i\int_0^\pi \frac{1}{2}\underbrace{\left\{\psi(\theta,2\pi)^2-\psi(\theta,0)^2\right\}}_{0}\sin(\theta)d\theta \\
&= 0
\end{aligned}$$

Since $\mathbf{n}$ is arbitrary, we arrive at $\mathfrak{o}_\psi = \mathbf{0}$. Next, we point out that the spherical harmonics are well-known to be eigenfunctions of the Laplace-Beltrami operator (see [17, §IV] for a detailed discussion), and for $\mathbb{S}^2$ they satisfy

$$-\Delta_{\mathbb{S}^2}Y_{\ell,k} = \ell(\ell+1)Y_{\ell,k}. \tag{5.16}$$

Standard arguments involving eigenfunction expansions yield

$$\int_{\mathbb{S}^2}\overline{\psi(\eta)}(-\Delta_{\mathbb{S}^2})\psi(\eta)d\omega(\eta) = \sum_{\ell=0}^{\infty}\sum_{k=1}^{2\ell+1}\ell(\ell+1)|\psi_{\ell,k}|^2. \tag{5.17}$$

Since $\mathfrak{o}_\psi = \mathbf{0}$, (5.17) and (5.11) taken together imply the second half of (5.13). ∎

For real-valued ψ, the quantity $\Delta\Omega$ is a measure of "ripple" or "angular frequency" in ψ. The two ways of writing $\Delta\Omega$ show this. In the first way, the expression (5.11), which gives $\Delta\Omega$ in terms of the expected value of the Laplace-Beltrami operator, shows the connection with "roughness" or rapid variation in ψ; such behavior will be magnified in $-\Delta_{\mathbb{S}^2}\psi$. Indeed, although in quantum mechanics or signal processing one usually thinks of "Δp" or "$\Delta\omega$" as being the spread in momentum or frequency, they can also be written as expected values for the Laplacian. The second way of expressing $\Delta\Omega$ is in the series form found in (5.13). It is clear from this form that a large value of $\Delta\Omega$ means a slow decay in the series coefficients in (5.13); thus, the higher order spherical harmonics are contributing significantly to ψ. Since the higher order spherical harmonics vary rapidly, their presence indicates high frequency behavior in ψ.

We now turn to finding expressions for $\Delta\Omega^2 = \mathrm{var}_\psi(\Omega)$ when ψ belongs to one of the sampling spaces defined earlier. As in §IV, we will again adopt the conventions used throughout the last half of §III, starting with those in (3.6). In particular, we will use $\psi\to f$. Our first result is the following.

COROLLARY 5.4: *Let $f \in \mathcal{V}$ and let $\mathbf{f} = (\, f_1 \quad \cdots \quad f_{N+M}\,)^T$ be the coordinate vector for f relative to the basis $\{G(p \cdot \xi_j)\}_{j=1}^{N+M}$. Then, if the coefficients f_j are real, and if G is twice continuously differentiable, we have*

$$\mathrm{var}_\psi(\Omega) = \sum_{j,k=1}^{N+M} f_j f_k \int_{\mathbb{S}^2} G(\eta \cdot \xi_j)(-\Delta_{\mathbb{S}^2})G(\eta \cdot \xi_k)d\omega(\eta), \tag{5.18}$$

Moreover, we also have that the integral appearing above has the form

$$\int_{\mathbb{S}^2} G(\eta \cdot p)(-\Delta_{\mathbb{S}^2})G(\eta \cdot q)d\omega(\eta) = \frac{d}{dt}\left[(t^2-1)\frac{d}{dt}(G \star G)\right]_{t\,=\,p\cdot q} \tag{5.19}$$

PROOF: Since f is real—its coefficients are real and G is real-valued—, $\mathfrak{o}_f = \mathbf{0}$. Express f in terms of the basis $\{G(p \cdot \xi_j)\}_{j=1}^{N+M}$, substitute the result into (5.11), taking account of the fact that $\mathfrak{o}_f = \mathbf{0}$, and collect terms. The final result is (5.18).

To obtain (5.19), recall that $G(\eta \cdot q)$ can be represented in terms of a series of spherical harmonics via (2.5). Fix q. Using (5.16) and (2.5), it is easy to show that

$$\begin{aligned}
(-\Delta_{\mathbb{S}^2})G(\eta \cdot q) &= \sum_{\ell=0}^{\infty} a_\ell \frac{4\pi}{2\ell+1} \sum_{j=1}^{2\ell+1} \left((-\Delta_{\mathbb{S}^2})Y_j^\ell(\eta)\right) \bar{Y}_j^\ell(q) \\
&= \sum_{\ell=0}^{\infty} a_\ell \frac{4\pi}{2\ell+1} \sum_{j=1}^{2\ell+1} \ell(\ell+1) Y_j^\ell(\eta)\bar{Y}_j^\ell(q) \\
&= \sum_{\ell=0}^{\infty} a_\ell \ell(\ell+1) P_\ell(3; \eta \cdot q).
\end{aligned} \tag{5.20}$$

Thus, we see that the integral $\int_{\mathbb{S}^2} G(\eta \cdot p)(-\Delta_{\mathbb{S}^2})G(\eta \cdot q)d\omega(\eta)$ is the star-product of G with $(-\Delta_{\mathbb{S}^2})G(\eta \cdot q)$, which is in fact an SBF in its own right. Combining (2.7) and (2.9) (with $m = 2$ in both) with (5.20), we have

$$\int_{\mathbb{S}^2} G(\eta \cdot p)(-\Delta_{\mathbb{S}^2})G(\eta \cdot q)d\omega(\eta) = \sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} \ell(\ell+1) a_\ell^2 P_\ell(3; p \cdot q)$$

Since [12, p. 141, eqn. 1] $[(t^2-1)P'(3;t)]' = \ell(\ell+1)P(3;t)$, it is easy to see that

$$\int_{\mathbb{S}^2} G(\eta \cdot p)(-\Delta_{\mathbb{S}^2})G(\eta \cdot q)d\omega(\eta) = \frac{d}{dt}\left[(t^2-1)\frac{d}{dt}\left(\sum_{\ell=0}^{\infty} \frac{4\pi}{2\ell+1} a_\ell^2 P_\ell(3;t)\right)\right]_{t\,=\,p\cdot q}$$

The series in parentheses above is the $m = 2$ expression for $G \star G$ give in (2.9). Replacing the series by $G \star G$ yields (5.19). ∎

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