# Notes on Special Functions

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## Introduction

These notes are for our classes on *special functions*. The elementary functions that appear in the first few semesters of calculus – powers of $x$, ln, sin, cos, exp, etc. are not enough to deal with the many problems that arise in science and engineering. *Special function* is a term loosely applied to additional functions that arise frequently in applications. We will discuss three of them here: Bessel functions, the gamma function, and Legendre polynomials.

## 1 Bessel Functions

### 1.1 A heat flow problem

Bessel functions come up in problems with circular or spherical symmetry. As an example, we will look at the problem of heat flow in a circular plate. This problem is the same as the one for diffusion in a plate. By adjusting our units of length and time, we may assume that the plate has a unit radius. Also, we may assume that certain physical constants involved are 1. In polar coordinates, the equation governing the temperature at any point $(r, \theta)$ and any time $t$ is the *heat equation*,

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,. \tag{1}$$

By itself, this equation only represents the energy balance in the plate. Two more pieces of information are necessary for us to be able to determine the temperature:

the initial temperature, $u(0, r, \theta) = f(r, \theta)$, and the temperature at the boundary, $u(t, 1, \theta)$, which we will assume to be 0. These are called the *initial conditions* and *boundary conditions*, respectively.

Solving these equations is done via a technique called separation of variables. In this technique, we simply ignore the initial conditions and work only with the heat equation and the boundary condition $u(t, 1, \theta) = 0$. We will not go through the whole procedure here. Instead, we will step in at the point where Bessel functions come into play. The idea is to look for a solution $u$ to (1) that satisfies the boundary condition $u(t, 1, \theta) = 0$ and that has the form

$$u(t, r, \theta) = e^{-\lambda t} R(r) \left\{ \begin{array}{c} \cos(n\theta) \\ \sin(n\theta) \end{array} \right. ,$$

where $n$ is an integer and is given. The constant $\lambda > 0$ and the function $R(r)$ are unknown. A solution having this form satisfies the boundary condition if and only if $R(1) = 0$. If we substitute $u$ into (1), then, with a little calculus and some algebra, we find that $R$ satisfies the ordinary differential equation,

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\lambda r^2 - n^2)R = 0\,. \tag{2}$$

Finally, we make a change of variables $x = \sqrt{\lambda}\, r$ and $y(x) = R(r)$. By the chain rule, we get $\frac{dy}{dx} = \frac{1}{\sqrt{\lambda}} \frac{dR}{dr}$ and $\frac{d^2y}{dx^2} = \frac{1}{\lambda} \frac{d^2R}{dr^2}$. Using these in the previous equation, we get this,

$$\underbrace{\lambda r^2}_{x^2} \frac{d^2y}{dx^2} + \underbrace{\sqrt{\lambda}\, r}_{x} \frac{dy}{dx} + (\underbrace{\lambda r^2}_{x^2} - n^2)y = 0\,,$$

from which it follows that

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0. \tag{3}$$

This is *Bessel's equation* and its solutions are *Bessel functions*.

## 1.2 Bessel's equation and the method of Frobenius

We are going to solve Bessel's equation (3) using a power series method developed in the nineteenth century by the German mathematician Ferdinand Frobenius. The usual power series method won't work because Bessel's equation has a singularity at $x = 0$. Instead, we will assume that there is a solution of the form

$$y(x) = \sum_{k=0}^{\infty} a_k x^{\beta+k}, \quad a_0 \neq 0, \tag{4}$$

where $\beta$ can be fractional and doesn't have to be a non-negative integer; $\beta$ is sometimes called an *index*.

We are going to substitute $y(x)$ into (3). Before we do that, we will put (3) into a form that's easier to use:

$$x\frac{d}{dx}\Big(x\frac{dy}{dx}\Big) + (x^2 - n^2)y = 0 \tag{5}$$

We need to calculate three series. The first is just this:

$$\begin{aligned}
x\frac{d}{dx}\Big(x\frac{dy}{dx}\Big) &= \sum_{k=0}^{\infty} a_k x\frac{d}{dx}\Big(x\frac{d}{dx}x^{\beta+k}\Big) \\
&= \sum_{k=0}^{\infty}(\beta+k)^2 a_k x^{\beta+k} \\
&= \beta^2 a_0 x^\beta + (\beta+1)^2 a_1 x^{\beta+1} + \cdots + (\beta+k)^2 a_k x^{\beta+k} + \cdots.
\end{aligned}$$

The second series is this one:

$$-n^2 y = -n^2 a_0 x^\beta - n^2 a_1 x^{\beta+1} - n^2 a_2 x^{\beta+2} - \cdots - n^2 a_k x^{\beta+k} - \cdots.$$

The final series is:

$$x^2 y = a_0 x^{\beta+2} + a_1 x^{\beta+3} + \cdots + a_{k-2} x^{\beta+k} + \cdots.$$

From (5), the sum of the three series is identically 0. Thus, we have that

$$\begin{aligned}
(\beta^2 - n^2)a_0 x^\beta &+ ((\beta+1)^2 - n^2)a_1 x^{\beta+1} + \{((\beta+2)^2 - n^2)a_2 + a_0\}x^{\beta+2} + \\
\cdots &+ \{((\beta+k)^2 - n^2)a_k + a_{k-2}\}x^{\beta+k} + \cdots \equiv 0\,.
\end{aligned}$$

As is the case for ordinary power series, the only way the series above can vanish identically is for the coefficient of each power to be 0. This implies that the following equations hold:

$$\begin{aligned}
(\beta^2 - n^2)a_0 &= 0 \quad \text{(Indicial equation)} && (6)\\
((\beta+1)^2 - n^2)a_1 &= 0 && (7)\\
((\beta+k)^2 - n^2)a_k + a_{k-2} &= 0,\ k \ge 2 \quad \text{(Recursion relation).} && (8)
\end{aligned}$$

The indicial equation (6) gives us that $\beta^2 = n^2$, since we assumed that $a_0 \neq 0$ at the start. Consequently, $\beta = \pm n$. We will take $\beta = n \ge 0$ for now. Later on, we will explain what happens if we take $\beta = -n$. Also, it is *not* necessary to assume that $n$ is an integer.

The next equation in our list is (7). With $n \geq 0$, the factor $(\beta+1)^2 - n^2 = (n+1)^2 - n^2 > 0$, so we may divide by it to get $a_1 = 0$.

The last equation (8) is the recursion relation. The coefficient of $a_k$ is positive, and so we can divide by it to put (8) in the form

$$\begin{aligned} a_k &= -\frac{a_{k-2}}{(n+k)^2 - n^2} \\ &= -\frac{a_{k-2}}{k(k+2n)}\,. \end{aligned} \tag{9}$$

If we put $k = 3$ in (9), then we get $a_3 = -\frac{a_1}{3(2n+3)} = 0$. A similar calculation shows that $a_5 = 0$, $a_7 = 0$, and that in general $a_{odd\ k} = 0$. To take care of the even $k$, replace $k$ by $2k$ in (9) and simplify to get

$$a_{2k} = -\frac{a_{2k-2}}{2^2 k(k+n)} \tag{10}$$

At this point, we iterate (10).

$$\begin{aligned} a_2 &= -\frac{a_0}{2^2 \cdot 1 \cdot (n+1)} \\ a_4 &= -\frac{a_2}{2^2 \cdot 2 \cdot (n+2)} = \frac{(-1)^2 a_0}{2^{2+2} \cdot 1 \cdot 2 \cdot (n+1)(n+2)} \\ a_6 &= -\frac{a_4}{2^2 \cdot 3 \cdot (n+3)} = \frac{(-1)^3 a_0}{2^{2 \cdot 3} 3!(n+1)(n+2)(n+3)} \\ &\vdots \\ a_{2k} &= -\frac{a_{2k-2}}{2^2 \cdot k \cdot (n+k)} = \frac{(-1)^k a_0}{2^{2k} k!(n+1)\cdots(n+k)} \end{aligned}$$

We now can write down the series for $y(x)$, which is

$$y(x) = a_0 x^n + \sum_{k=1}^{\infty} \frac{(-1)^k a_0 x^{2k+n}}{2^{2k} k!(n+1)\cdots(n+k)} \tag{11}$$

We point out that this series is actually a valid solution to Bessel's equation for *any* value of $n$, except for negative integers. The Bessel functions we will define below are simply ones with a particular choice of the constant $a_0$. When

$n = 0, 1, 2, \ldots$, the constant $a_0$ is chosen to be $a_0 = 1/(2^n n!)$. Using this and again doing a little algebra, we can put $y$ in the form

$$J_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} \tag{12}$$

The function $J_n(x)$ is called the order $n$ Bessel function of the first kind. Of course, it solves (3). For non-integer $n$, a similar expression holds, except that $(n+k)!$ gets replaced by a gamma function, $\Gamma(n+k+1)$,

$$J_n(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(n+k+1)} \left(\frac{x}{2}\right)^{2k+n} \tag{13}$$

We will discuss the gamma function later. For now, we note that if $m \geq 0$ is an integer, then $\Gamma(m+1) = m!$ .

## 1.3 Orthogonal series of Bessel Functions

Recall that in discussing the heat flow problem in section 1.1, we arrived at an eigenvalue problem in which we needed to find a function $R(r)$ that satisfies (2), the condition $R(1) = 0$, and is continuous at $r = 0$. We then reduced that problem to solving Bessel's equation (3), with $y(x) = R(r)$ and $x = \sqrt{\lambda}\, r$. Now, what we have seen is that $y = J_n(x)$, which is defined by the series in (12), solves (3). Also, for any integer $n \geq 0$, the solution behaves like $x^n$ near $x = 0$, so it is continuous there. We thus have that $R(r) = J_n(\sqrt{\lambda}\, r)$ satisfies two of three conditions on $R$. The last one is that $R(1) = 0$, and it will be satisfied if $J_n(\sqrt{\lambda}) = 0$ – i.e., if $\sqrt{\lambda} > 0$ is a *positive* zero of $J_n$. (For all $n > 0$, $J_n(0) = 0$.)

The first question one ought to ask here is whether there are *any* positive zeros for $J_n$. The answer is, yes. Indeed, there are infinitely many of them. We will not prove this here. We will denote the positive zeros of $J_n$ by $0 < j_{n,1} < j_{n,2} < \cdots < j_{n,m} < \cdots$. That is, $j_{n,m}$ is the $m^{th}$ positive zero of $J_n$, counting from small to large.

We want to discuss orthogonality properties of $R_m(r) := J_n(j_{n,m} r)$. The usual way to do this is to go back to (2) and work with that equation and various integrals. Rather than do that, we want to take a different approach here, one that directly uses inner products. We will first need the following lemma.

**Lemma 1.1** *If $V$ is a (real) vector space with an inner product $\langle \cdot\,, \cdot \rangle$, and if $L : V \to V$ satisfies $\langle L(\mathbf{u}), \mathbf{v} \rangle = \langle \mathbf{u}, L(\mathbf{v}) \rangle$, then the eigenvectors of $L$ corresponding to distinct eigenvalues are orthogonal.*

**Proof:** Suppose that $\mathbf{u}_1$ and $\mathbf{u}_2$ are eigenvectors corresponding to the eigenvalues $\lambda_1$ and $\lambda_2$, respectively. We assume that these are distinct, so $\lambda_1 \neq \lambda_2$. First, note that $\langle L(\mathbf{u}_1), \mathbf{u}_2\rangle = \langle \lambda_1\mathbf{u}_1, \mathbf{u}_2\rangle = \lambda_1\langle \mathbf{u}_1, \mathbf{u}_2\rangle$. Similarly, we get $\langle \mathbf{u}_1, L(\mathbf{u}_2)\rangle = \lambda_2\langle \mathbf{u}_1, \mathbf{u}_2\rangle$. By our assumption on $L$, we have $\langle L(\mathbf{u}_1), \mathbf{u}_2\rangle = \langle \mathbf{u}_1, L(\mathbf{u}_2)\rangle$, so $\lambda_1\langle \mathbf{u}_1, \mathbf{u}_2\rangle = \lambda_2\langle \mathbf{u}_1, \mathbf{u}_2\rangle$, which gives us that

$$(\lambda_1 - \lambda_2)\langle \mathbf{u}_1, \mathbf{u}_2\rangle = 0.$$

Since $\lambda_1 \neq \lambda_2$, we can divide by $\lambda_1 - \lambda_2$ to get $\langle \mathbf{u}_1, \mathbf{u}_2\rangle = 0.$, which, by definition, means that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal. □

When an operator $L$ satisfies the conditions of the lemma, we say that it is *self adjoint* with respect to the inner product. To apply the lemma to the case at hand, we need to properly choose the inner product and the operator $L$. The inner product we will use is

$$\langle f, g\rangle = \int_0^1 f(r)g(r)r dr,$$

and the operator $L$ will be

$$L(u) = \frac{1}{r}\Big(\frac{n^2}{r}u - ru'' - u'\Big) = \frac{1}{r}\Big(\frac{n^2}{r}u - (ru')'\Big)$$

and we will take $V$ to be the space of twice continuously differentiable functions $u$ that vanish at $r = 1$ and, for $n > 0$, at $r = 0$ as well. In any event, we have this chain of equations:

$$\begin{aligned}
\langle L(u), v\rangle - \langle u, L(v)\rangle &= \int_0^1 \frac{r}{r}\Big\{\frac{n^2uv}{r} - (ru')'v - \frac{n^2uv}{r} + u(rv')'\Big\}dr \\
&= \int_0^1 \Big\{-(ru')'v + u(rv')'\Big\}dr \\
&= ruv' - rv'u\Big|_0^1 + \int_0^1 \underbrace{ru'v' - ru'v'}_{0}\,dr \quad \text{(By parts.)} \\
&= 1\cdot(\underbrace{u(1)}_{0}v'(1) - u'(1)\underbrace{v(1)}_{0}) - 0\cdot(u(0)v'(0) - u'(0)v(0)) \\
&= 0
\end{aligned}$$

The point is that $L$ defined above is self adjoint for the given inner product. Now, the eigenvectors for $L$ are just the functions $R_m(r) = J_n(j_{n,m}r)$ and the eigenvalues are $\lambda_m = j_{n,m}^2$, which are distinct for distinct $m$'s. Applying the lemma to this case gives us the orthogonality of the $J_n(j_{n,m}r)$'s.

**Theorem 1.2** *If $m \neq m'$, then $\int_0^1 J_n(j_{n,m}r)J_n(j_{n,m'}r)rdr = 0$.*

With some effort, we could compute $\|R_m\|^2 = \int_0^1 J_n(j_{n,m}r)^2 rdr$. Instead, we will simply state the result, which is $\int_0^1 J_n(j_{n,m}r)^2 rdr = \frac{1}{2}J_{n+1}(j_{n,m})^2$. If we now define the functions

$$u_{n,m}(r) = \frac{\sqrt{2}\, J_n(j_{n,m}r)}{J_{n+1}(j_{n,m})}, \;\; m = 1, 2, \ldots,$$

we have an orthonormal family of Bessel functions that is, $\langle u_{n,m}, u_{n,m'} \rangle = \delta_{m,m'}$. These can be used to expand functions in the same way as Fourier series.

# 2 The Gamma Function

The gamma function is an extension of factorials to other numbers besides non-negative integers. It will allow us to make sense out of expressions like $(3/2)!$, $(-1/3)!$, and so on. We will start by defining this function, which is denoted by $\Gamma(x)$, for all $x > 0$.

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt\,. \tag{14}$$

We now wish to establish two very important properties of the gamma function. The first is an identity similar to $(n+1)! = (n+1)\cdot n!$, and the second relates factorials and gamma functions.

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x), \;\; x > 0. &(15)\\ \Gamma(n+1) &= n!\,, \;\; n = 0, 1, \ldots\,. &(16) \end{aligned}$$

For the first, we perform these steps:

$$\begin{aligned} \Gamma(x+1) &= \int_0^\infty t^x \underbrace{e^{-t}dt}_{d(-e^{-t})} \\ &= -t^x e^{-t}\Big|_0^\infty + \int_0^\infty \frac{d}{dt}\Big(t^x\Big)e^{-t}dt \\ &= x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma(x) \end{aligned}$$

To get the second, use the identity we proved, along with $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$, this way. We have $\Gamma(2) = 1\cdot\Gamma(1) = 1\cdot 1 = 1$. Also, $\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot 1 = 2!$. Similarly, $\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot 2! = 3!$. Repeating this gives us $\Gamma(n+1) = n!$.

There is one other thing that we need to mention. If we rewrite (15) as

$$\Gamma(x) = \Gamma(x+1)/x,$$

then the right side makes sense for $-1 < x < 0$. This enables us to define $\Gamma(x)$ for such $x$. Indeed, we can also write the previous formula as

$$\Gamma(x) = \Gamma(x+2)/(x(x+1)),$$

which allows us to define $\Gamma(x)$ for $-2 < x < -1$. In this way, we obtain values for $\Gamma(x)$ for all real $x$, except $x = 0, -1, -2$, etc. Finally, using (15), it is easy to see that for any $x$ that is not 0 or a negative integer, we have

$$\frac{\Gamma(x+k+1)}{\Gamma(x+1)} = (x+1)(x+2)\cdots(x+k).$$

This identity was implicitly used in defining Bessel functions having non-integer order. Recall that in section 1.2 we chose $a_0 = 1/(2^n n!)$ in the integer case. For the non-integer case, we can choose $a_0 = 1/(2^n\Gamma(n+1))$ to get the same effect.

# 3 Legendre Polynomials

Legendre polynomials come up in connection with three dimensional problems having spherical symmetry. The simplest situation that gives rise to them is solving for the steady-state temperature $u(r, \varphi, \varphi)$ in a sphere of radius 1. (We will use the convention that $\varphi$ is the colatitude, which angle measured off of the $z$-axis, and $\theta$ is the azimuthal angle, effectively the longitude.) The temperature $u$ satisfies Laplace's equation, $\nabla^2 u = 0$. In spherical coordinates, this equation is

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial u}{\partial\varphi}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 u}{\partial\theta^2} = 0\,. \qquad (17)$$

As in the case of the heat equation in the plate, we seek special solutions to this equation, ones of the form $u = r^\ell F(\varphi)$. Note that we are assuming that $u$ is independent of the longitude $\theta$. Also, we do not assume anything about $\ell$, other than that it is real and satisfies $\ell \geq 0$. (We need $\ell \geq 0$ to make $u$ continuous at $r = 0$.) Substituting $u$ into (17) and doing a little calculus and algebra, we arrive the equation

$$\frac{1}{\sin\varphi}\frac{d}{d\varphi}\left(\sin\varphi\frac{dF}{d\varphi}\right) + \ell(\ell+1)F(\varphi) = 0\,.$$

If we change variables in this equation to $x = \cos\varphi$ and $y(x) = F(\varphi)$, then, after some work, the equation above becomes Legendre's equation,

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right) + \ell(\ell+1)y = 0\,. \tag{18}$$

Here again we are faced with an eigenvalue problem. To be defined and continuous on the whole sphere, the original function $F(\varphi)$ has to be continuous at $\varphi = 0$ (north pole) and $\varphi = \pi$ (south pole). Thus, $y$ should be continuous at $x = \cos 0 = 1$ and $x = \cos\pi = -1$.

We now want to look at the eigenvalue problem in which $y$ satisfies (18) and is twice continuously differentiable on the interval $[-1, 1]$. We will apply Lemma 1.1 to the following situation: Take $V = C^{(2)}[-1, 1]$, and use the differential operator and inner product below.

$$\begin{aligned} L(y) &:= \frac{d}{dx}\left(\left(1-x^2\right)\frac{dy}{dx}\right) \\ \langle f, g\rangle &:= \int_{-1}^{1} f(x)g(x)dx. \end{aligned} \tag{19}$$

We leave it as an exercise to show that $L$ is self adjoint; that is, that $\langle L(u), v\rangle = \langle u, L(v)\rangle$.

Recall that when we discussed eigenvalue problems, we looked at the operator $L$ in connection with polynomials. In particular, we observed that if $n$ is a non-negative integer, then $L : \mathcal{P}_n \to \mathcal{P}_n$. Moreover the matrix of $L$ relative to $B = \{1, x, \ldots, x^n\}$ was the matrix

$$A = \begin{pmatrix} -0\cdot 1 & 0 & 2\cdot 1 & 0 & \cdots & 0 \\ 0 & -1\cdot 2 & 0 & 3\cdot 2 & \cdots & 0 \\ 0 & 0 & -2\cdot 3 & 0 & \cdots & 0 \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -(n-1)n & 0 \\ 0 & 0 & 0 & 0 & \cdots & -n(n+1) \end{pmatrix}.$$

Because this matrix is upper triangular, its eigenvalues are its diagonal entries; these are all of the form $-k(k+1)$, for $k = 0, 1, \ldots, n$. Each eigenvalue has a corresponding eigenvector whose entries are coefficients for a polynomial $P_k(x)$. For instance, $k = 2$, which gives the eigenvalue $-6$, has the eigenvector $(-1\; 0\; 3\; 0\; \cdots\; 0)^T$ and corresponds to $P_2(x) = 3x^2 - 1$. Indeed, each $k$ then gives us an eigenvalue

$-k(k+1)$ and a polynomial $P_k(x)$ that satisfies $L(P_k) = -k(k+1)P_k$ and that has degree $k$. Since polynomials are infinitely differentiable in $x$, $P_k$ belongs to the space $V = C^{(2)}[-1,1]$. Consequently, the $P_k$'s are eigenfunctions of $L$ corresponding to the eigenvalue $-k(k+1)$. Moreover, Lemma 1.1 thus implies that these satisfy $\langle P_k, P_{k'} \rangle = 0$ when $k \neq k'$.

The point is that, up to constant multiples, these polynomials are exactly the same as the ones generated by the Gram-Schmidt process applied to $\{1, x, x^2, \ldots\}$, with the inner product being the one in (19). As for the eigenvalue problem (18), with $y \in V$, it turns out that the only eigenvalues are those with $\ell = k$, $k = 0, 1, \ldots$, and the only eigenfunctions are the polynomials $P_k$, which are called the Legendre polynomials.

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