STABILITY RESULTS FOR SCATTERED-DATA INTERPOLATION ON EUCLIDEAN SPHERES

by

F. J. Narcowich, N. Sivakumar and J. D. Ward

Center for Approximation Theory

Department of Mathematics

Texas A&M University

College Station, TX 77843–3368

USA

Abstract

Let S^m denote the unit sphere in $\mathbf{R}^{m+1}$ and d_m the geodesic distance in S^m. A *spherical-basis function* approximant is a function of the form

$$s(x) = \sum_{j=1}^{M} a_j \varphi(d_m(x, x_j)), \qquad x \in S^m,$$

where $(a_j)_1^M$ are real constants, $\varphi : [0, \pi] \to \mathbf{R}$ is a fixed function, and $(x_j)_1^M$ is a set of distinct points in S^m. It is known that if φ is a *strictly positive definite function in* S^m, then the *interpolation matrix* $(\varphi(d_m(x_j, x_k)))_{j,k=1}^M$ is positive definite, hence invertible, for every choice of distinct points $(x_j)_1^M$ and every positive integer M. The paper studies a salient subclass of such functions φ, and provides stability estimates for the associated interpolation matrices.

§1. Introduction

Let $\varphi : [0, \infty) \to \mathbf{R}$ be a fixed function. A radial-basis function (RBF) approximant is a function of the form

$$s(x) = \sum_{j=1}^{M} a_j \varphi(\|x - x_j\|), \qquad x \in \mathbf{R}^d, \tag{1.1}$$

where $(a_j)_1^M$ are real constants, $(x_j)_1^M$ are distinct points in $\mathbf{R}^d$, and $\|\cdot\|$ denotes the Euclidean norm in $\mathbf{R}^d$. Such functions provide a useful tool for multivariate approximation because of their ability to provide multidimensional surface fits using scattered data. Radial-basis functions have been actively researched over the last decade, resulting in a rather substantial body of literature; we refer the reader to the informative surveys [Bu, D, P].

A central problem in the theory of RBF interpolation is the existence of interpolants of the form (1.1): Given arbitrary real numbers $c_1, \ldots c_M$ and a set $(x_j)_1^M$ of distinct points in $\mathbf{R}^d$, is there a function s of the form (1.1) such that the interpolation conditions $s(x_k) = c_k$ are satisfied, for every $1 \leq k \leq M$? The success of the RBF method is largely due to the fact that there exists a large family of functions φ for which the answer to this question is in the affirmative. In fact, it is known that for several classes of functions φ, the *interpolation matrix*

$$A_\varphi := (\varphi(x_j - x_k))_{j,k=1}^M \tag{1.2}$$

is invertible for *every* choice of distinct points $(x_j)_1^M$ and every positive integer M [Mi, MN1-2]. It is manifest that the invertiblity of A_φ guarantees the existence of a unique function s of the form (1.1) satisfying the interpolation conditions.

The qualitative results described above answer the key question of existence of RBF interpolants, but they do not shed light on another important and related issue: stability of the interpolation matrix and its inverse, and hence that of the underlying interpolation scheme. Understanding this issue requires quantitative results, and divers results of this nature have been obtained in the literature; e.g., [B, Ba, BSW, JS, NW1-2, NSW, S, Su].

The subject matter of the current paper is a close cousin of the RBF interpolation problem. Let S^m denote the unit sphere in $\mathbf{R}^{m+1}$, and let d_m denote the geodesic distance on S^m. Considerable interest is now being evinced in the interpolation of scattered data on S^m; see, for instance, the engaging survey [C] and [F1-3, M1-2, RS, Sch, XC]. One way of constructing the required interpolants is via *spherical-basis function* (SBF) approximants, *viz.,* functions of the form

$$s(x) = \sum_{j=1}^{M} a_j \varphi(d_m(x, x_j)), \qquad x \in S^m, \qquad a_j \in \mathbf{R}, \tag{1.3}$$

where φ is a fixed real-valued function on $[0, \pi]$, and $(x_j)_1^M$ are distinct points in S^m. The analogy between (1.3) and (1.1) is evident, and, as in the case of RBF interpolation, the crux of the matter here is the determination of useful classes of functions φ for which the interpolation matrix

$$A_\varphi := (\varphi(d_m(x_j, x_k)))_{j,k=1}^M \tag{1.4}$$

is invertible for every choice of distinct points $(x_j)_1^M$ in S^m and every M. The best-known class is that of *strictly positive definite functions in* S^m, to wit, functions φ for which the matrices A_φ are positive definite for every choice of distinct points $(x_j)_1^M$ in S^m; as in the case of the RBF theory, this function-class is rooted in Schoenberg's seminal work on positive definite functions and metric geometry.

The extant literature on SBF interpolation deals primarily with qualitative aspects of the theory, and the purpose of this paper is to provide some quantitative results. Specifically, we wish to obtain stability estimates for interpolation matrices of the form (1.4), for functions φ belonging to an important subclass of strictly positive definite function in S^m.

The paper is organized as follows. The main results are given in the next section, which is in two parts. The first part focuses exclusively on interpolation by means of a specific family of functions which has its origins in univariate spline theory. These results are then used in the second part of the section to derive stability estimates for interpolation matrices associated with more general functions. The third and final section of the paper considers interpolation on the unit circle. Not unexpectedly, stronger results are obtained in this case. In particular, it is shown that some of the estimates there are best possible, whilst some others are close to being so. The paper is interspersed with examples and remarks which serve to illustrate the general results.

§2. Main results

Let S^m denote the unit sphere in $\mathbf{R}^{m+1}$, and d_m the geodesic distance in S^m; that is, if $x, y \in S^m$, then

$$d_m(x, y) := \cos^{-1}\langle x, y\rangle, \tag{2.1}$$

where $\langle x, y\rangle$ is the inner product of x and y (here and elsewhere we shall identify points in $\mathbf{R}^{m+1}$ with their position vectors). Note that $d_m(x, y)$ is also the angle subtended by the vectors x and y at the origin.

Let $\{e_1, \ldots, e_{m+1}\}$ be the standard unit-vector basis for $\mathbf{R}^{m+1}$. If $x \in S^m$, then x can be expressed as

$$x = t\, e_{m+1} + \sqrt{1 - t^2}\, \eta\,, \qquad -1 \le t \le 1, \qquad t := \langle e_{m+1}, x\rangle, \tag{2.2}$$

where η is a unit vector spanned by $e_1, \ldots, e_m$. Equivalently,

$$x = (\cos\phi)\, e_{m+1} + (\sin\phi)\, \eta\,, \qquad 0 \le \phi \le \pi, \tag{2.3}$$

where

$$\cos\phi = \langle e_{m+1}, x\rangle, \qquad i.e., \qquad \phi = d_m(e_{m+1}, x). \tag{2.4}$$

Lemma 2.1. *Let $0 \le \phi_0 \le \pi$ and let $C(\phi_0)$ denote any cap of S^m with colatitude ϕ_0; that is,*

$$C(\phi_0) := \{x \in S^m \,:\, d_m(x, \hat{x}) \le \phi_0\},$$

for some fixed point $\hat{x}$ on S^m, called the centre of the cap. The surface area of $C(\phi_0)$ is given by

$$\omega_{m-1} \int_0^{\phi_0} \sin^{m-1}\phi\, d\phi, \tag{2.5}$$

where $\omega_{m-1} := [2\pi^{m/2}/\Gamma(m/2)]$ is the surface area of the unit sphere S^{m-1} in $\mathbf{R}^m$.

Proof. We may assume that the cap is centred at $e_{m+1} = (0, 0, \ldots, 0, 1)$. The surface-area element $d\omega_m$ on S^m is given by the equation (cf. [Mü, p. 1])

$$d\omega_m = (\sin^{m-1}\phi)\, d\phi\, d\omega_{m-1}, \tag{2.6}$$

where ϕ is given by (2.3) and (2.4). Since $C(\phi_0)$ is centred at e_{m+1}, ϕ ranges from 0 to ϕ_0, so the surface area of $C(\phi_0)$ is given by the integral

$$\int_{S^{m-1}} \int_0^{\phi_0} \sin^{m-1}\phi\, d\phi\, d\omega_{m-1}, \tag{2.7}$$

whence the result. ■

Let n be a positive integer and let M_n denote the centred cardinal B-spline of order n, to wit, M_n is the n-fold convolution of the characteristic function of the interval $[-1/2, 1/2]$ with itself. We recall that M_n is a piecewise polynomial of degree $n-1$ satisfying the following conditions (see, for example, [Sc2]):

(P1) $M_n(\cdot) = M_n(-\cdot)$;

(P2) $M_n(x) > 0$ if $|x| < n/2$;

(P3) $M_n(x) = 0$ if $|x| \ge n/2$;

(P4) $\widehat{M_n}(\xi) = \left[\dfrac{\sin(\xi/2)}{(\xi/2)}\right]^n, \qquad \xi \in \mathbf{R}$.

Our main tool in the subsequent analysis will be the *B-spline symbol*, that is, the function σ_n defined by

$$\sigma_n(u) := \sum_{k\in\mathbf{Z}} M_n(k) e^{-iku}, \qquad -\pi \le u \le \pi. \tag{2.7}$$

Some relevant properties of the symbol are listed in the theorem below.

Theorem 2.2. *Let $n \geq 2$ be an integer. The following hold:*

(i) σ_n is an even trigonometric polynomial (i.e., a cosine polynomial) of degree $\lceil n/2 \rceil - 1$;

(ii) $\sigma_n(u) > 0$ for every $u \in [-\pi, \pi]$;

(iii) σ_n is a decreasing function on the interval $[0, \pi]$;

(iv) $\sigma_n(0) = 1$ and

$$\sigma_n(\theta) \leq \frac{\pi^2}{4} \left[\frac{\sin(\theta/2)}{(\theta/2)} \right]^n, \qquad 0 < \theta \leq \pi. \tag{2.8}$$

Proof. (i) Since M_n is an even function and $M_n(k) = 0$ for every $k \geq \lceil n/2 \rceil$, we have

$$\sigma_n(u) = M_n(0) + 2 \sum_{k=1}^{\lceil n/2 \rceil - 1} M_n(k) \cos(ku), \qquad u \in [-\pi, \pi]. \tag{2.9}$$

Assertions (ii) and (iii) have been proved by Schoenberg in [Sc3].

To prove (iv) we invoke Poisson's summation formula, which allows us to express σ_n as follows:

$$\sigma_n(u) = \sum_{k \in \mathbf{Z}} \widehat{M}_n(u + 2\pi k) = \sum_{k \in \mathbf{Z}} \left[\frac{\sin(\pi k + (u/2))}{(\pi k + (u/2))} \right]^n, \qquad -\pi \leq u \leq \pi. \tag{2.10}$$

Putting $u = 0$ in the second sum in (2.10) provides the equation $\sigma_n(0) = 1$. Moreover, (2.10) also reveals that

$$\sigma_n(\theta) = [\sin(\theta/2)]^n \sum_{k \in \mathbf{Z}} \frac{(-1)^{kn}}{[\pi k + (\theta/2)]^n} \leq [\sin(\theta/2)]^n \sum_{k \in \mathbf{Z}} \frac{1}{|\pi k + (\theta/2)|^n}, \quad 0 < \theta \leq \pi. \tag{2.11}$$

Now the right-hand side of (2.11) can be rewritten as

$$\begin{aligned} \left[\frac{\sin(\theta/2)}{(\theta/2)} \right]^n \left[1 + \sum_{k \neq 0} \left| \frac{\theta}{2\pi k + \theta} \right|^n \right] &= \left[\frac{\sin(\theta/2)}{(\theta/2)} \right]^n \left[1 + \sum_{k=1}^{\infty} \left(\frac{\theta}{2\pi k + \theta} \right)^n + \sum_{k=1}^{\infty} \left(\frac{\theta}{2\pi k - \theta} \right)^n \right] \\ &=: \left[\frac{\sin(\theta/2)}{(\theta/2)} \right]^n \left[1 + \sum_{k=1}^{\infty} f_{1,k}(n, \theta) + \sum_{k=1}^{\infty} f_{2,k}(n, \theta) \right]. \end{aligned} \tag{2.12}$$

The functions $f_{i,k}(n, u)$ increase with u in $[0, \pi]$ for every positive integer k and $i = 1, 2$. Hence $f_{i,k}(n, \theta) \leq f_{i,k}(n, \pi)$ for every k and i. Finally,

$$\begin{aligned} 1 + \sum_{k=1}^{\infty} f_{1,k}(n, \pi) + \sum_{k=1}^{\infty} f_{2,k}(n, \pi) &= 1 + \sum_{k=1}^{\infty} \frac{1}{(2k+1)^n} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^n} \\ &= 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^n} \leq 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{4}, \end{aligned} \tag{2.13}$$

and this, taken in conjunction with (2.12) and (2.11), finishes the proof. ∎

Definition 2.2. Let $0 < q \le \pi$ be a fixed number. We say that a finite subset $X = (x_j)_1^M$ of S^m is *q-separated* if $\min_{j \ne k} d_m(x_j, x_k) = q$.

The following theorem, which concerns scattered-data interpolation on S^m by means of B-spline symbols, is the first of two main results of this section. It will provide us with a key tool for the treatment, in the second part of the section, of interpolation problems involving a more general class of functions.

Theorem 2.3. *Let $0 < q \le \pi$ be fixed and let $n \ge 2$ be an integer. Define*

$$\kappa(m, q) := \max \left\{ 4m \left(\frac{\pi^2}{q} \right)^{m-1}, 5^m \left(\frac{\pi}{2} \right)^{m-1} \right\}. \tag{2.14}$$

If M is any positive integer and $(x_j)_1^M$ is any q-separated set in S^m, then

$$\begin{aligned} \left[\sigma_n(0) - \frac{\pi}{q} \kappa(m,q) \, \sigma_n(q) \right] \left[\sum_{j=1}^{M} |c_j|^2 \right] &\le \sum_{j,k=1}^{M} c_j c_k \sigma_n(d_m(x_j, x_k)) \\ &\le \left[\sigma_n(0) + \frac{\pi}{q} \kappa(m,q) \, \sigma_n(q) \right] \left[\sum_{j=1}^{M} |c_j|^2 \right], \end{aligned} \tag{2.15}$$

for every vector $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$.

Proof. Following an argument of Ingham [I] (also used subsequently in [NW1, p. 79]), we find that the quadratic form
$\sum_{j,k=1}^M c_j c_k \sigma_n(d_m(x_j, x_k))$ is bounded above and below, respectively, by the quantities

$$(\sigma_n(0) \pm \gamma_M) \, \|c\|_2^2, \tag{2.16}$$

where

$$\gamma_M := \max_k \left[{\sum_{j=1}^{M}}' \sigma_n(d_m(x_j, x_k)) \right], \tag{2.17}$$

with Σ' denoting the sum over all j such that $j \ne k$. Let k_0 be the index where the maximum in (2.17) is attained, that is,

$$\gamma_M = {\sum_{j=1}^{M}}' \sigma_n(d_m(x_j, x_{k_0})), \tag{2.18}$$

with Σ' meaning the obvious. We may assume that $x_{k_0} = e_{m+1}$, the 'north pole'. Let $N := \lfloor (\pi/q) \rfloor$, and partition the set $X' := \{x_j \,:\, 1 \le j \le M,\ j \ne k_0\}$ as follows:

$$X' = \left(\bigcup_{k=1}^{N-2} R_k \right) \cup C_{N-1}, \tag{2.19}$$

where

$$R_k := \{x_j \,:\, kq \le d_m(x_j, x_{k_0}) < (k+1)q\} \tag{2.20}$$

and

$$C_{N-1} := \{x_j \,:\, (N-1)q \le d_m(x_j, x_{k_0}) \le \pi\}\,. \tag{2.21}$$

Our first task is to estimate the cardinalities of the sets R_k and C_{N-1}.

Suppose $1 \le k \le N-2$ and let card (R_k) denote the cardinality of R_k. We borrow a 'packing' argument from [NW1, p. 80] and apply it, *mutatis mutandis*, to spheres. Since the set (x_j) is q-separated, a cap with colatitude $q/2$ centred at any x_j is essentially disjoint from such a cap centred at any other x_k, $j \ne k$. Therefore

$$\operatorname{card}(R_k) \le \frac{S_1(k,q)}{S_2(q)}, \tag{2.22}$$

where $S_2(q)$ is the surface area of a cap with colatitude $q/2$ and $S_1(k,q)$ is the surface area of the 'ring' in S^m which lies between the $(k-\frac{1}{2})\,q^{th}$ and $(k+\frac{3}{2})\,q^{th}$ 'parallels'. In other words, $S_1(k,q)$ is the surface area of the (set) difference of the cap of colatitude $(k+\frac{3}{2})\,q$ centred at x_{k_0} (the north pole) and the concentric cap of colatitude $(k-\frac{1}{2})\,q$. So $S_1(k,q)$ is the difference between the surface area of the first (and bigger) cap and that of the second, and this number, in view of Lemma 2.1, is given by the formula

$$S_1(k,q) = \omega_{m-1}\int_{(k-\frac{1}{2})q}^{(k+\frac{3}{2})q} \sin^{m-1}\phi\, d\phi. \tag{2.23}$$

Lemma 2.1 also tells us that

$$S_2(q) = \omega_{m-1}\int_0^{q/2} \sin^{m-1}\phi\, d\phi, \tag{2.24}$$

and using these last two equations in (2.22) provides the estimate

$$\operatorname{card}(R_k) \le \frac{\int_{(k-\frac{1}{2})q}^{(k+\frac{3}{2})q} \sin^{m-1}\phi\, d\phi}{\int_0^{q/2} \sin^{m-1}\phi\, d\phi}, \qquad 1 \le k \le N-2. \tag{2.25}$$

We shall now bound the numerator of the right-hand side of (2.25) from above, and the denominator from below.

Since $\sin\phi \le \phi$, we see that

$$\int_{(k-\frac{1}{2})q}^{(k+\frac{3}{2})q} \sin^{m-1}\phi\, d\phi \le \int_{(k-\frac{1}{2})q}^{(k+\frac{3}{2})q} \phi^{m-1}\, d\phi = \frac{[q_2^m - q_1^m]}{m}, \tag{2.26}$$

where $q_1 := (k-\frac{1}{2})\,q$ and $q_2 := (k+\frac{3}{2})\,q$. Now $q_2^m - q_1^m \le m\,\pi^{m-1}\,(q_2 - q_1)$ because $q_1 < q_2 = (k+\frac{3}{2})\,q \le (N-\frac{1}{2})\,q < \pi$. Consequently

$$\int_{(k-\frac{1}{2})q}^{(k+\frac{3}{2})q} \sin^{m-1}\phi\, d\phi \le \pi^{m-1}\,(q_2 - q_1) = 2\pi^{m-1}q, \qquad 1 \le k \le N-2, \tag{2.27}$$

by (2.26).

As for the denominator of the right-hand side of (2.25), we find that

$$\int_0^{q/2} \sin^{m-1}\phi \, d\phi \geq \left(\frac{2}{\pi}\right)^{m-1} \int_0^{q/2} \phi^{m-1} \, d\phi = \frac{q^m}{2m\pi^{m-1}}, \tag{2.28}$$

where we have used the familiar estimate $\sin\phi \geq (2\phi/\pi)$, $0 \leq \phi \leq \pi/2$, which is valid here because $q \leq \pi$.

Using (2.28) and (2.27) in (2.25), we arrive at

$$\operatorname{card}(R_k) \leq \frac{4m\pi^{2m-2}}{q^{m-1}}, \qquad 1 \leq k \leq N-2. \tag{2.29}$$

Next we consider the set C_{N-1}. An upper bound for its cardinality is provided by

$$\operatorname{card}(C_{N-1}) \leq \frac{S_3(N,q)}{S_2(q)}, \tag{2.30}$$

where $S_2(q)$ is the same as before and $S_3(N,q)$ is the surface area of the cap

$$\{x \in S^m \,:\, (N-(3/2))\,q \leq d_m(x, x_{k_0}) \leq \pi\} = \{x \in S^m \,:\, 0 \leq d_m(x, -x_{k_0}) \leq \pi - (N-(3/2))\,q\}. \tag{2.31}$$

From Lemma 2.1, (2.28), and the inequality $\sin\phi \leq \phi$, we obtain

$$\begin{aligned} \operatorname{card}(C_{N-1}) \leq \frac{\int_0^{\pi-(N-(3/2))q} \sin^{m-1}\phi \, d\phi}{\int_0^{q/2} \sin^{m-1}\phi \, d\phi} &\leq \frac{\int_0^{\pi-(N-(3/2))q} \phi^{m-1} \, d\phi}{\int_0^{q/2} \sin^{m-1}\phi \, d\phi} \\ &\leq \frac{(1/m)\,(\pi-(N-(3/2))\,q)^m}{(1/(2m))\,(1/\pi)^{m-1}\,q^m}. \end{aligned} \tag{2.32}$$

Now $(\pi - N + (3/2))\,q < 5q/2$ because $(N+1)\,q > \pi$, whence

$$\operatorname{card}(C_{N-1}) \leq 2\pi^{m-1}\,(5/2)^m, \tag{2.33}$$

on account of (2.32).

Inequalities (2.29) and (2.33) may now be used to estimate the quantity γ_M of (2.18):

$$\begin{aligned} \gamma_M &= \sum_{k=1}^{N-2} \sum_{x_j \in R_k} \sigma_n(d_m(x_j, x_{k_0})) + \sum_{x_j \in C_{N-1}} \sigma_n(d_m(x_j, x_{k_0})) \\ &\leq \sum_{k=1}^{N-2} \left(\frac{4m\pi^{2m-2}}{q^{m-1}}\right) \sigma_n(kq) + 2\pi^{m-1}(5/2)^m\, \sigma_n((N-1)\,q), \end{aligned} \tag{2.34}$$

where we have used the fact that σ_n decreases on the interval $[0,\pi]$ (see Theorem 2.2 (iii)). Using this fact one more time, and remembering the definition of $\kappa(m,q)$ from (2.14), along with the fact that $(N-1)\,q<\pi$, we obtain the following chain of inequalities from (2.34):

$$\gamma_M \le \kappa(m,q) \sum_{k=1}^{N-1} \sigma_n(kq) \le \kappa(m,q)\,(N-1)\,\sigma_n(q) \le \kappa(m,q)\,(\pi/q)\,\sigma_n(q). \tag{2.35}$$

The required result obtains directly from (2.35) and (2.16). ∎

Corollary 2.4. *Let $\kappa(m,q)$ be defined by (2.14). The following hold:*

(i) Let $q\in(0,\pi]$ and let $n\ge 2$ be a positive integer. If M is any positive integer and $(x_j)_1^M$ is any q-separated set in S^m, then we have the bounds

$$\left[1-(\pi^3/(4q))\,\kappa(m,q)\left(\frac{\sin(q/2)}{(q/2)}\right)^n\right]\left[\sum_{j=1}^M |c_j|^2\right] \le \sum_{j,k=1}^M c_j c_k \sigma_n(d_m(x_j,x_k)) \le \left[1+(\pi^3/(4q))\,\kappa(m,q)\left(\frac{\sin(q/2)}{(q/2)}\right)^n\right]\left[\sum_{j=1}^M |c_j|^2\right], \tag{2.36}$$

for every vector $c=(c_1,\ldots,c_M)\in\mathbf{R}^M$.

(ii) Let q, M and $(x_j)_1^M$ be as above. There exists a positive integer $K=K(q)$ such that

$$\sum_{j,k=1}^M c_j c_k \sigma_n(d_m(x_j,x_k)) \ge \left[1-(\pi^3/(4q))\,\kappa(m,q)\left(\frac{\sin(q/2)}{(q/2)}\right)^n\right]\left[\sum_{j=1}^M |c_j|^2\right] > 0, \tag{2.37}$$

for every nonzero vector $c=(c_1,\ldots,c_M)\in\mathbf{R}^M$, and every $n>K$.

Proof. Assertion (i) is a consequence of Theorem 2.3 and Theorem 2.2 (iv), whereas (ii) follows from (i) and the fact that $[\sin(q/2)/(q/2)]^n$ tends to zero as n tends to infinity. ∎

Thus far we have dealt exclusively with interpolation by means of B-spline symbols. We shall now consider a more general class of functions.

Let $P_l(m+1;t)$, $l\ge 0$, denote the $(m+1)$-dimensional Legendre polynomial of degree l. We recall (see, for example, [Mü, p. 16]) that $P_l(m+1;t)$ is an algebraic polynomial of degree l in t, and that the set of functions $\{P_l\}_{l=0}^\infty$ is an orthogonal family with respect to the weight function $(1-t^2)^{(m-2)/2}$ on the interval $[-1,1]$. Suppose $n\ge 2$ is an integer. We know from Theorem 2.2 (i) that the B-spline symbol σ_n is a cosine polynomial of degree $\lceil n/2\rceil-1$. We may therefore write σ_n in the form

$$\sigma_n(u)=\tilde{\sigma_n}(\cos u), \qquad u\in[-\pi,\pi], \tag{2.38}$$

where $\tilde{\sigma_n}(t)$ is an algebraic polynomial of degree $\lceil n/2 \rceil - 1$ in the variable t. In view of the aforesaid orthogonality property of the Legendre polynomials, every algebraic polynomial of degree $\lceil n/2 \rceil - 1$ admits a 'Legendre expansion' in terms of the functions $P_l(m+1;\cdot)$, $0 \le l \le \lceil n/2 \rceil - 1$. In particular, there exist real numbers $\alpha_l(m;n)$, $0 \le l \le \lceil n/2 \rceil - 1$, such that

$$\sum_{l=0}^{\lceil n/2 \rceil - 1} |\alpha_l(m;n)| > 0 \tag{2.39}$$

and

$$\sigma_n(u) = \tilde{\sigma_n}(\cos u) = \sum_{l=0}^{\lceil n/2 \rceil - 1} \alpha_l(m;n)\, P_l(m+1;\cos u), \qquad -\pi \le u \le \pi. \tag{2.40}$$

For sake of convenience, we set $\alpha_l(m;n) = 0$ for every $l \ge \lceil n/2 \rceil$ and write

$$\sigma_n(u) = \tilde{\sigma_n}(\cos u) = \sum_{l=0}^{\infty} \alpha_l(m;n)\, P_l(m+1;\cos u), \qquad -\pi \le u \le \pi. \tag{2.41}$$

Let g be a continuous real-valued function on the interval $[0,\pi]$ which admits a representation of the form

$$g(u) := \sum_{l=0}^{\infty} a_l\, P_l(m+1;\cos u), \qquad u \in [0,\pi], \tag{2.42}$$

where

$$a_l > 0 \quad \forall\, l \ge 0 \qquad \text{and} \qquad \sum_{l=0}^{\infty} a_l < \infty. \tag{2.43}$$

Thanks to [Sc1] and [XC], we know that g induces a strictly positive definite function on S^m, *viz.*, the matrix $(g(d_m(x_j,x_k)))_{j,k=1}^M$ is positive definite for every choice of distinct points $(x_j)_1^M$ in S^m and every M. We note in passing that the representation of g in [Sc1] and [XC] is given in terms of ultraspherical (Gegenbauer) polynomials rather than Legendre polynomials. The equivalent formulation (2.42) has been used in [N] and [NW3], and will better suit our purposes here.

Lemma 2.5. *Suppose g is a continuous real-valued function on $[0,\pi]$, representable in the form (2.42). The following hold:*

(i) If $x,y \in S^m$, then

$$g(d_m(x,y)) = \sum_{l=0}^{\infty} a_l \frac{\omega_m}{d_{m,l}} \sum_{j=1}^{d_{m,l}} Y_j^l(x) Y_j^l(y), \tag{2.44}$$

where ω_m is the surface area of S^m and $d_{m,l}$ is the dimension of the space of the order-l spherical harmonics $\{Y_j^l\}$ on S^m.

(ii) Let $(x_k)_1^M$ be any finite set of distinct points in S^m. If $c = (c_1,\ldots,c_M) \in \mathbf{R}^M$, then

$$\sum_{r,s=1}^{M} c_r c_s g(d_m(x_r,x_s)) = \sum_{l=0}^{\infty} a_l \frac{\omega_m}{d_{m,l}} \sum_{j=1}^{d_{m,l}} \left| \sum_{k=1}^{M} c_k Y_j^l(x_k) \right|^2. \tag{2.45}$$

Proof. (i) This is derived on page 3 of [NW3]. For future reference we record here that

$$d_{m,l} = \begin{cases} \frac{(2l+m-1)\,(l+m-2)!}{l!\,(m-1)!}, & \text{if } l \geq 1; \\ 1, & \text{if } l = 0. \end{cases} \tag{2.46}$$

(ii) This is an easy consequence of (i). ∎

Proposition 2.6. *Suppose g is a continuous real-valued function on $[0, \pi]$, representable in the form (2.42). Suppose $n \geq 2$ is an integer. Let σ_n be given by (2.7), and the sequence $\{\alpha_l(m; n)\}_{l=0}^{\infty}$ via equation (2.41). Define*

$$\Gamma(m; n) := \min_{0 \leq l \leq \lceil n/2 \rceil - 1} \left\{ \frac{a_l}{|\alpha_l(m; n)|} \, : \, \alpha_l(m; n) \neq 0 \right\}. \tag{2.47}$$

If $(x_k)_1^M$ is any set of distinct points in S^m and $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$, then

$$\sum_{r,s=1}^{M} c_r c_s g(d_m(x_r, x_s)) \geq \Gamma(m; n) \sum_{r,s=1}^{M} c_r c_s \sigma_n(d_m(x_r, x_s)). \tag{2.48}$$

Proof. The definition of $\Gamma(m; n)$ and the positivity of a_l for every nonnegative integer l ensure that

$$a_l \geq \Gamma(m; n)\, \alpha_l(m; n), \qquad \forall\, l \geq 0. \tag{2.49}$$

Consequently, Lemma 2.5 provides the relations

$$\begin{aligned} \sum_{r,s=1}^{M} c_r c_s g(d_m(x_r, x_s)) &= \sum_{l=0}^{\infty} a_l \frac{\omega_m}{d_{m,l}} \sum_{j=1}^{d_{m,l}} \left| \sum_{k=1}^{M} c_k Y_j^l(x_k) \right|^2 \\ &\geq \Gamma(m; n) \sum_{l=0}^{\infty} \alpha_l(m; n) \frac{\omega_m}{d_{m,l}} \sum_{j=1}^{d_{m,l}} \left| \sum_{k=1}^{M} c_k Y_j^l(x_k) \right|^2 \\ &= \Gamma(m; n) \sum_{r,s=1}^{M} c_r c_s \sigma_n(d_m(x_r, x_s)). \end{aligned} \tag{2.50}$$

∎

The next theorem is the second of two main results of this section.

Theorem 2.7. *Let g be as in the preceding proposition. Suppose $q \in (0, \pi]$ is fixed, and let $\kappa(m, q)$ be defined by (2.14). Let $N = N(q) \geq 2$ be the smallest integer such that*

$$1 - (\pi^3/(4q))\, \kappa(m, q) \left(\frac{\sin(q/2)}{(q/2)} \right)^N > 0, \tag{2.51}$$

and let $\alpha_l(m; N)$, $l \geq 0$, and $\Gamma(m; N)$ be defined by equations (2.41) and (2.47), respectively.

(i) If $X = (x_k)_1^M$ is any q-separated subset of S^m and $A_g(X) := (g(d_m(x_j, x_k)))_{j,k=1}^M$, then

$$\sum_{r,s=1}^{M} c_r c_s g(d_m(x_r, x_s)) \geq \Gamma(m; N) \left[1 - (\pi^3/(4q))\, \kappa(m, q) \left(\frac{\sin(q/2)}{(q/2)}\right)^N\right] \left[\sum_{k=1}^{M} |c_k|^2\right] \tag{2.52}$$

for every vector $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$. In particular, the spectral norm of $(A_g(X))^{-1}$ may be estimated thusly:

$$\|(A_g(X))^{-1}\|_2 \leq \frac{1}{\Gamma(m; N) \left[1 - (\pi^3/(4q))\, \kappa(m, q) \left(\frac{\sin(q/2)}{(q/2)}\right)^N\right]}. \tag{2.53}$$

(ii) The spectral norm of $A_g(X)$ satisfies the following inequality:

$$\|A_g(X)\|_2 \leq \frac{2m\, \pi^{m-1} \left(\omega_m/\omega_{m-1}\right) \left[\sum_{l=0}^{\infty} a_l\right]}{q^m}. \tag{2.54}$$

(iii) The spectral condition number of $A_g(X)$ can be bounded as follows:

$$\begin{aligned} \mathit{cond}_2(A_g(X)) &:= \|A_g(X)\|_2 \, \|(A_g(X))^{-1}\|_2 \\ &\leq \frac{2m\, \pi^{m-1} \left(\omega_m/\omega_{m-1}\right) \left[\sum_{l=0}^{\infty} a_l\right]}{\Gamma(m; N)\, q^m \left[1 - (\pi^3/(4q))\, \kappa(m, q) \left(\frac{\sin(q/2)}{(q/2)}\right)^N\right]}. \end{aligned} \tag{2.55}$$

Proof. (i) The estimate (2.52) follows by combining Proposition 2.6 with Corollary 2.4 (ii), whilst (2.53) is consequent upon (2.52) and standard linear algebra.

(ii) Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_M$ be the eigenvalues of $A_g(X)$. Then

$$\|A_g(X)\|_2 = \lambda_M < \sum_{k=1}^{M} \lambda_k = \mathrm{trace}(A_g(X)) = M g(0) = M \left[\sum_{l=0}^{\infty} a_l\right], \tag{2.56}$$

the last equation resulting from (2.42) and the fact that $P_l(m+1; 1) = 1$ for every l [Mü, p. 16]. Since the set $(x_k)_1^M$ is q-separated, a cap of radius $q/2$ centred at any x_j is essentially disjoint from such a cap centred at any other x_k, $k \neq j$. It follows that

$$M \text{ (surface area of a cap of radius } q/2) \leq \omega_m, \tag{2.57}$$

whence

$$M\, \omega_{m-1} \int_0^{q/2} \sin^{m-1} \phi \, d\phi \leq \omega_m, \tag{2.58}$$

using (2.5). The required result now obtains from (2.58), (2.28), and (2.56).

(iii) This follows at once from (2.53) and (2.54). ∎

We remark that the relations (2.56) also appear in [LLS] and [M3].

In addition to the given number q and the explicitly defined quantity $\kappa(m,q)$, the estimate (2.53) is also patently dependent upon the numbers N and $\Gamma(m;N)$. The following remark shows how these numbers can be estimated, if not optimally, at least practically.

Remark 2.8. (i) Define

$$t(m,q) := \frac{2q}{\pi^3 \kappa(m,q)}, \tag{2.59}$$

and observe that

$$0 < t(m,q) < 1 \tag{2.60}$$

(because $0 < q \le \pi$). Let N be the smallest positive integer such that

$$N \ge 24\,[1 - (\pi^2/80)]^{-1} \left[\frac{\log(1/t(m,q))}{q^2}\right]. \tag{2.61}$$

It follows from (2.61) that

$$N \ge \frac{\log(1/t(m,q))}{(q^2/24)\,(1-(q^2/80))}, \tag{2.62}$$

whence we obtain

$$-(q^2/24) + (q^4/1920) \le (1/N)\,\log(t(m,q)). \tag{2.63}$$

Since $(\sin x)/x \le 1 - (x^2/6) + (x^4/24)$ for $x > 0$ and $t^{1/N} = \exp((\log t)/N) \ge 1 + (1/N)\,\log t$ for $0 < t < 1$, we conclude from (2.63) that

$$\frac{\sin(q/2)}{(q/2)} \le (t(m,q))^{1/N}, \tag{2.64}$$

that is,

$$\left[\frac{\sin(q/2)}{(q/2)}\right]^N \le t(m,q). \tag{2.65}$$

It is easy to verify that (2.65) supplies the estimate

$$1 - (\pi^3/(4q))\,\kappa(m,q)\left(\frac{\sin(q/2)}{(q/2)}\right)^N \ge 1/2. \tag{2.66}$$

(ii) Recall from (2.40) that

$$\sigma_N(u) = \tilde{\sigma}_N(t) = \sum_{l=0}^{\lceil N/2\rceil - 1} \alpha_l(m;N) P_l(m+1;t), \quad t = \cos u, \qquad u \in [-\pi,\pi]. \tag{2.67}$$

Since the polynomials $\{P_l\}$ are orthogonal with respect to the weight $w(t) := (1-t^2)^{(m-2)/2}$ on the interval $[-1,1]$, a standard argument shows that

$$\alpha_l(m;N) = \frac{1}{\|P_l\|_w^2} \int_{-1}^{1} \tilde{\sigma}_N(t)\, P_l(m+1;t)\, w(t)\, dt, \qquad 0 \le l \le \lceil N/2\rceil - 1, \tag{2.68}$$

where

$$\|f\|_w^2 := \int_{-1}^{1} |f(t)|^2 w(t)\, dt. \tag{2.69}$$

Since $\tilde{\sigma}_N(t) = \sigma_N(\cos^{-1} t)$ is positive and less than 1 for every t in $[-1,1]$ (Theorem 2.2, (ii) – (iv)), an application of the Cauchy-Schwarz inequality (in $(L^2[-1,1], \|\cdot\|_w)$) yields the estimate

$$|\alpha_l(m;N)| \le \frac{\left(\int_{-1}^{1} w(t)\, dt\right)^{1/2}}{\|P_l\|_w} =: \frac{c_1(m)}{\|P_l\|_w}, \qquad 0 \le l \le \lceil N/2 \rceil - 1. \tag{2.70}$$

From [Mü, p. 15], we learn that

$$\|P_l\|_w = \sqrt{\frac{\omega_m}{\omega_{m-1}}}\, d_{m,l}^{-1/2}, \qquad 0 \le l \le \lceil N/2 \rceil - 1, \tag{2.71}$$

where $d_{m,l}$ is given by (2.46). Using (2.71) in (2.70) leads to the inequality

$$|\alpha_l(m;N)| \le c_2(m)\, d_{m,l}^{1/2}, \qquad 0 \le l \le \lceil N/2 \rceil - 1, \tag{2.72}$$

where $c_2(m) := c_1(m)\, (\omega_m/\omega_{m-1})^{-1/2}$. Recall from (2.46) that

$$d_{m,l} = \begin{cases} \frac{(2l+m-1)\,(l+m-2)\cdots(l+1)}{(m-1)!}, & \text{if } l \ge 1; \\ 1, & \text{if } l = 0. \end{cases} \tag{2.73}$$

Since $\prod_{j=1}^{m-2}(l+j) \le l^{m-1}\,(m-1)!$ and $2l + m - 1 \le l(m+1)$ for $l \ge 1$, we see from (2.73) that

$$d_{m,l} \le \begin{cases} (m+1)\, l^{m-1} & \text{if } l \ge 1; \\ 1, & \text{if } l = 0. \end{cases} \tag{2.74}$$

Using (2.74) in (2.72) provides the following:

$$|\alpha_l(m;N)| \le \begin{cases} c_3(m)\, l^{(m-1)/2} & \text{if } l \ge 1; \\ c_2(m), & \text{if } l = 0, \end{cases} \tag{2.75}$$

where $c_3(m) := \sqrt{(m+1)}\, c_2(m)$. Thus, from (2.75) and (2.47) we deduce that

$$\Gamma(m;N) \ge (c_4(m))^{-1} \min\left\{a_0, \min_{1 \le k \le \lceil N/2 \rceil - 1} \frac{a_k}{k^{(m-1)/2}}\right\}, \tag{2.76}$$

where $c_4(m) := \max\{c_2(m), c_3(m)\}$.

We conclude the section with an example.

Example 2.9. Let g, X and $A_g(X)$ be as in Theorem 2.7. If N is chosen according to (2.56) and the sequence $(a_l)_0^\infty$ is monotone decreasing, we see from (2.53), (2.66), and (2.76) that

$$\|(A_g(X))^{-1}\|_2 = O(N^{(m-1)/2}\, a_{\lceil N/2 \rceil}^{-1}). \tag{2.77}$$

In particular, if $a_l \sim (1+l^\alpha)^{-1}$ for some fixed $\alpha > 1$, then

$$\|(A_g(X))^{-1}\|_2 = O\left[(\log(1/t(m,q))/q^2)^{\alpha+(m-1)/2}\right], \qquad q \to 0^+, \tag{2.78}$$

where $t(m,q)$ is given by (2.59). Since $\kappa(m,q) = 4m\,(\pi^2/q)^{m-1}$ for small q (see (2.14)), (2.78) may be expressed as follows:

$$\|(A_g(X))^{-1}\|_2 = O\left[(q^{-2}\,\log(q^{-m}))^{\alpha+(m-1)/2}\right], \qquad q \to 0^+. \tag{2.79}$$

§3. Interpolation on the unit circle

In this section we shall deal with interpolation on S^1, a case in which, perhaps not surprisingly, more information can be extracted. Indeed, we shall find that it is possible to obtain estimates which, in many cases, are either optimal or nearly so.

Let $n \geq 2$ be an integer and recall that σ_n is the trigonometric polynomial given by

$$\sigma_n(u) = M_n(0) + 2\sum_{k=1}^{\lceil n/2\rceil-1} M_n(k)\cos(ku) = M_n(0) + 2\sum_{k=1}^{\infty} M_n(k)\cos(ku), \qquad u \in [-\pi,\pi]. \tag{3.1}$$

Let $g : [0,\pi] \to \mathbf{R}$ be a continuous function of the form

$$g(u) = \sum_{l=0}^{\infty} a_l \cos(lu), \qquad u \in [0,\pi], \tag{3.2}$$

with

$$a_l > 0 \text{ for all } l \geq 0 \quad \text{and} \quad \sum_{l=0}^{\infty} a_l < \infty. \tag{3.3}$$

We know from [Sc1] and [XC] that g induces a strictly positive definite function on S^1, meaning that the matrix $(g(d_1(x_j,x_k)))_{j,k=1}^M$ is positive definite for every choice of distinct points $(x_k)_1^M$ in S^1 and every positive integer M.

Lemma 3.1. *Suppose g is a function of the type described above, and let $(x_j)_1^M$ be a set of distinct points in S^1. Assume that $x_j = (\cos\theta_j, \sin\theta_j)$, $1 \leq j \leq M$, where $0 \leq \theta_j < 2\pi$. Then*

$$\begin{aligned}\sum_{j,k=1}^{M} c_j c_k g(d_1(x_j,x_k)) &= \sum_{l=0}^{\infty} a_l \sum_{j,k=1}^{M} c_j c_k \cos(l(\theta_j-\theta_k)) \\ &= \sum_{l=0}^{\infty} a_l \left[\left(\sum_{k=1}^{M} c_k\cos(l\theta_k)\right)^2 + \left(\sum_{k=1}^{M} c_k\sin(l\theta_k)\right)^2\right],\end{aligned} \tag{3.4}$$

for every vector $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$.

Proof. See [XC]. ∎

Theorem 3.2. *Let g be as above and let $n \geq 2$ be an integer. Define*

$$\Gamma_n := \min \left\{ \frac{a_0}{M_n(0)}, \min_{1 \leq k \leq \lceil n/2 \rceil - 1} \frac{a_k}{2M_n(k)} \right\}. \tag{3.5}$$

If $(x_k)_1^M$ is any set of distinct points in S^1, then

$$\sum_{j,k=1}^{M} c_j c_k g(d_1(x_j, x_k)) \geq \Gamma_n \sum_{j,k=1}^{M} c_j c_k \sigma_n(d_1(x_j, x_k)), \tag{3.6}$$

for every vector $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$.

Proof. The definition of Γ_n, positivity of every a_l and the fact that $M_n(k) = 0$ for every $k \geq \lceil n/2 \rceil$ combine to show that

$$a_l \geq \Gamma_n \, M_n^{(1)}(l), \qquad l \geq 0, \tag{3.7}$$

where

$$M_n^{(1)}(0) = M_n(0) \quad \text{and} \quad M_n^{(1)}(l) = 2M_n(l), \quad l \geq 1. \tag{3.8}$$

So Lemma 3.1 implies

$$\begin{aligned} \sum_{j,k=1}^{M} c_j c_k g(d_1(x_j, x_k)) &= \sum_{l=0}^{\infty} a_l \left[\left(\sum_{k=1}^{M} c_k \cos(l\theta_k) \right)^2 + \left(\sum_{k=1}^{M} c_k \sin(l\theta_k) \right)^2 \right] \\ &\geq \Gamma_n \sum_{l=0}^{\infty} M_n^{(1)}(l) \left[\left(\sum_{k=1}^{M} c_k \cos(l\theta_k) \right)^2 + \left(\sum_{k=1}^{M} c_k \sin(l\theta_k) \right)^2 \right] \\ &= \Gamma_n \sum_{j,k=1}^{M} c_j c_k \sigma_n(d_1(x_j, x_k)). \end{aligned} \tag{3.9}$$

∎

The following remark gives a simple estimate for the quantity Γ_n.

Remark 3.3. Since $M_n(0) + 2\sum_{k=1}^{\lceil n/2 \rceil - 1} M_n(k) = \sigma_n(0) = 1$ (see Theorem 2.2, (i) and (iv)), we deduce that each of the (positive) quantities $M_n(0)$ and $2M_n(k)$, $1 \leq k \leq \lceil n/2 \rceil - 1$, is smaller than 1. Therefore

$$\Gamma_n \geq \min\{a_k \,:\, 0 \leq k \leq \lceil n/2 \rceil - 1\}. \tag{3.10}$$

Combining Theorem 3.2 with earlier results yields the following:

Theorem 3.4. *Suppose g is as in Theorem 3.2, $q \in (0, \pi]$, and $(x_j)_1^M$ is any q-separated set in S^1. Let $A := g(d_1(x_j, x_k))_{j,k=1}^M$, and let $N = N(q) \geq 2$ be the smallest integer such that*

$$1 - \frac{5\pi^3}{4q} \left(\frac{\sin(q/2)}{(q/2)} \right)^N > 0. \tag{3.11}$$

The following hold:

(i)

$$\sum_{j,k=1}^M c_j c_k g(d_1(x_j, x_k)) \geq \Gamma_N \left[1 - \frac{5\pi^3}{4q} \left(\frac{\sin(q/2)}{(q/2)} \right)^N \right] \|c\|_2^2, \tag{3.12}$$

where Γ_N is defined by (3.5) and $c = (c_1, \ldots, c_M)$ is any vector in $\mathbf{R}^M$. In particular, the spectral norm of A^{-1} satisfies the estimate

$$\|A^{-1}\|_2 \leq \frac{1}{\Gamma_N \left[1 - \frac{5\pi^3}{4q} \left(\frac{\sin(q/2)}{(q/2)} \right)^N \right]}. \tag{3.13}$$

(ii) The spectral norm of A satisfies the following estimate:

$$\|A\|_2 \leq \frac{2\pi \left[\sum_{l=0}^\infty a_l \right]}{q}. \tag{3.14}$$

(iii) The spectral condition number of A can be bounded as follows:

$$\mathit{cond}_2(A) \leq \frac{2\pi \left[\sum_{l=0}^\infty a_l \right]}{q\, \Gamma_N \left[1 - \frac{5\pi^3}{4q} \left(\frac{\sin(q/2)}{(q/2)} \right)^N \right]}. \tag{3.15}$$

Proof. (i) The lower bound for the quadratic form comes from the first inequality in (3.6) and that in (2.36); one has merely to note that $\kappa(m, q)$, as defined in (2.14), equals 5 when $m = 1$.

(ii) This comes from putting $m = 1$ in (2.54) and recalling the values of ω_m and ω_{m-1} from the statement of Lemma 2.1.

(iii) Combine (3.13) and (3.14). ∎

The following remark ought to be compared with Remark 2.8 and Example 2.9 from the previous section.

Remark 3.5. (i) A lower bound for N, although not necessarily the optimal one, is obtained by putting $m = 1$ in (2.59):

$$N \geq 24\,(1 - (\pi^2/80))^{-1} \frac{\log(5\pi^3/(2q))}{q^2}. \tag{3.16}$$

In analogy with (2.66) then, inequality (3.16) provides the estimate

$$1 - (5\pi^3/(4q)) \left(\frac{\sin(q/2)}{(q/2)} \right)^N \geq 1/2. \tag{3.17}$$

(ii) Let g and A be as in the preceding theorem. Combining (3.17) with Remark 3.3, we find that

$$\|A^{-1}\|_2 \le \frac{2}{\min\{a_k \,:\, 0 \le k \le \lceil N/2 \rceil - 1\}}. \tag{3.18}$$

In particular, if $(a_l)_0^\infty$ is a monotone decreasing sequence, then

$$\|A^{-1}\|_2 = O((a_{\lceil N/2 \rceil})^{-1}). \tag{3.19}$$

(iii) Let A be as above, and let $a_l \sim (1 + l^\alpha)^{-1}$ for some fixed $\alpha > 1$. An application of (3.19) and (3.16) reveals that

$$\|A^{-1}\|_2 = O\left[(q^{-2}\, \log(1/q))^\alpha\right], \qquad q \to 0^+. \tag{3.20}$$

This estimate is not sharp, however, and can be improved considerably (see Remark 3.8).

Our next pair of results concerns a special class of functions g. Of course this class is covered by Theorem 3.4, but our intention here is to present a different approach which provides tighter estimates in this special case. We shall see that some of these estimates are optimal.

Theorem 3.6. *Suppose*

$$g(u) := \sum_{l=0}^{\infty} \frac{\cos(lu)}{l^n + 1}, \qquad u \in [0, \pi], \tag{3.21}$$

where $n \ge 2$ is an integer. Let $0 < q \le \pi$ and define

$$\tilde{\Gamma}_{n,q} := \min\left\{ \frac{n}{2q}, \frac{q^{n-1}}{8n^{n-1}} \right\}. \tag{3.22}$$

Then

$$\sum_{j,k=1}^{M} c_j c_k g(d_1(x_j, x_k)) \ge (2\pi\, M_n(0)\, \tilde{\Gamma}_{n,q})\, (\sum_{j=1}^{M} |c_j|^2), \tag{3.23}$$

for every q-separated set $(x_j)_1^M$ in S^1 and every vector $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$.

Proof. Let

$$M_{n,q}(x) := M_n\left(\frac{nx}{2q}\right) \quad \text{and} \quad \varphi_{n,q}(u) := 2\pi \sum_{j \in \mathbf{Z}} M_{n,q}(u + 2\pi j). \tag{3.24}$$

Poisson's summation formula assures us that

$$\varphi_{n,q}(u) = \sum_{k \in \mathbf{Z}} \widehat{M_{n,q}}(k) e^{iku}, \tag{3.25}$$

and a simple calculation demonstrates that

$$\widehat{M_{n,q}}(k) = \frac{2q}{n}\, \widehat{M}_n\left(\frac{2qk}{n}\right) = \frac{2q}{n}\, \left[\frac{\sin(qk/n)}{(qk/n)}\right]^n. \tag{3.26}$$

Since $\widehat{M_{n,q}}(k) = \widehat{M_{n,q}}(-k)$, we may write

$$\varphi_{n,q}(u) = \widehat{M_{n,q}}(0) + 2\sum_{k=1}^{\infty} \widehat{M_{n,q}}(k)\cos(ku) =: \sum_{k=0}^{\infty} \alpha_k \cos(ku). \tag{3.27}$$

Setting

$$a_k := \frac{1}{k^n + 1}, \qquad k \geq 0, \tag{3.28}$$

we find, via the estimate $|\sin x| \leq 1$, that

$$a_0 = \frac{n}{2q}\,\alpha_0 \qquad \text{and} \qquad a_k \geq \frac{q^{n-1}}{8n^{n-1}}\,\alpha_k, \qquad k \geq 1, \tag{3.29}$$

whence

$$a_k \geq \tilde{\Gamma}_{n,q}\,\alpha_k, \qquad k \geq 0. \tag{3.30}$$

Proceeding now as in the proof of Theorem 3.2, we arrive at the inequality

$$\sum_{j,k=1}^{M} c_j c_k g(d_1(x_j, x_k)) \geq \tilde{\Gamma}_{n,q} \sum_{j,k=1}^{M} c_j c_k \varphi_{n,q}(d_1(x_j, x_k)). \tag{3.31}$$

Since $M_n(t) = 0$ whenever $|t| \geq n/2$, we see that

$$M_{n,q}(u + 2\pi j) = M_n\left(\frac{(u+2\pi j)n}{2q}\right) = 0, \qquad |u| \leq \pi, \quad j \neq 0. \tag{3.32}$$

Consequently,

$$\varphi_{n,q}(u) = 2\pi M_{n,q}(u) = 2\pi M_n\left(\frac{nu}{2q}\right), \qquad -\pi \leq u \leq \pi. \tag{3.33}$$

If $q \leq u \leq \pi$, then $\varphi_{n,q}(u) = 0$, again because $M_n(t) = 0$ for $t \geq n/2$. Thus

$$\sum_{j,k=1}^{M} c_j c_k \varphi_{n,q}(d_1(x_j, x_k)) = \varphi_{n,q}(0)\sum_{j=1}^{M} |c_j|^2 = 2\pi M_n(0)\sum_{j=1}^{M} |c_j|^2, \tag{3.34}$$

because $q \leq d_1(x_j, x_k) \leq \pi$ if $j \neq k$. The desired result follows from (3.34) and (3.31). ∎

Corollary 3.7. *Suppose $g(u) := \sum_{l=0}^{\infty} a_l \cos(lu)$, where $\sum_{l=0}^{\infty} a_l < \infty$ and $a_l \geq K\,(1 + l^{\alpha})^{-1}$, $l \geq 0$, for some fixed positive numbers α and K. Let n be the smallest positive integer such that $n \geq \max\{2, \alpha\}$. Let $0 < q \leq \pi$ and define $\tilde{\Gamma}_{n,q}$ as in Theorem 3.6. The following hold:*

(i)

$$\sum_{j,k=1}^{M} c_j c_k g(d_1(x_j, x_k)) \geq K\,(2\pi\, M_n(0)\,\tilde{\Gamma}_{n,q})\left(\sum_{j=1}^{M} |c_j|^2\right), \tag{3.35}$$

for every q-separated set $(x_j)_1^M$ in S^1 and every vector $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$.

(ii) If $A := (g(d_1(x_j, x_k)))_{j,k=1}^M$*, where* $(x_j)_1^M$ *is any* q*-separated set in* S^1*, then*

$$\|A^{-1}\|_2 \le (K\, 2\pi\, M_n(0)\, \tilde{\Gamma}_{n,q})^{-1}. \tag{3.36}$$

Proof. (i) Our choice of n makes it clear that $a_l \ge K\,(1 + l^n)^{-1} =: K\,\tilde{a}_l$ for every non-negative integer l. A (now-familiar) argument involving Lemma 3.1 (see the proof of Theorems 3.2 and 3.6) establishes the inequality

$$\sum_{j,k=1}^M c_j c_k g(d_1(x_j, x_k)) \ge K \sum_{j,k=1}^M c_j c_k \tilde{g}(d_1(x_j, x_k)), \tag{3.37}$$

where $\tilde{g}(u) := \sum_{l=0}^\infty \tilde{a}_l \cos(lu)$. The required result follows at once from Theorem 3.6.

(ii) This is a consequence of (i). ∎

Remark 3.8. (i) Let $a_l \sim (1 + l^\alpha)^{-1}$ for some fixed $\alpha > 1$, and let $n := \max\{2, \lceil\alpha\rceil\}$. If $A := (g(d_1(x_j, x_k)))_{j,k=1}^M$, where $(x_j)_1^M$ is any q-separated set in S^1, then Theorem 3.6 shows that $\|A^{-1}\|_2 \le (2\pi\, M_n(0)\, \tilde{\Gamma}_{n,q})^{-1}$. Now if q is small enough (less than $\pi/2$, for example), then $\tilde{\Gamma}_{n,q} = q^{n-1}/(8n^{n-1})$, so

$$\|A^{-1}\|_2 = O(q^{1-n}), \qquad q \to 0^+, \tag{3.38}$$

and this estimate is manifestly better than the one given in (3.20).

(ii) If $\alpha = \lceil\alpha\rceil = n \ge 2$ and $q \sim 1/M$, then (3.38) reads as follows:

$$\|A^{-1}\|_2 = O(M^{\alpha-1}), \qquad M \to \infty. \tag{3.39}$$

This estimate is best possible insofar as it concerns the dependence on M (see Example 3.13).

(iii) Theorem 3.4(ii) asserts that

$$\|A\|_2 = O(q^{-1}), \qquad q \to 0^+, \tag{3.40}$$

where $A = (g(d_1(x_j, x_k)))_{j,k=1}^M$ and $(x_k)_1^M$ is any q-separated set. In particular, if $q \sim 1/M$, then

$$\|A\|_2 = O(M), \qquad M \to \infty, \tag{3.41}$$

and this estimate is also sharp in the same sense as above (see Example 3.13).

We close this section by considering interpolation at equally spaced points on the unit circle, and using DFT methods to compute the spectral condition numbers of the associated interpolation matrices.

Let g be a generic function of the type considered hitherto in this section, that is, a continuous real-valued function on the interval $[0, \pi]$ satisfying the conditions laid out in (3.2) and (3.3).

Lemma 3.9. *Suppose g is as above and let $(x_j)_1^M$ be a set of distinct points in S^1. Assume that $x_j = (\cos\theta_j, \sin\theta_j)$, $1 \le j \le M$, where $0 \le \theta_j < 2\pi$. Then*

$$\sum_{j,k=1}^{M} c_j c_k g(d_1(x_j, x_k)) = \sum_{j,k=1}^{M} c_j c_k \left[\sum_{l\in\mathbf{Z}} b_l e^{il(\theta_j-\theta_k)}\right] = \sum_{l\in\mathbf{Z}} b_l \left|\sum_{k=1}^{M} c_k e^{il\theta_k}\right|^2, \tag{3.40}$$

where $c = (c_1, \ldots, c_M) \in \mathbf{R}^M$ and the sequence $\{b_l\}_{l\in\mathbf{Z}}$ is defined as follows:

$$b_0 := a_0, \qquad b_l := a_l/2, \quad l \ge 1, \qquad b_l := a_{-l}/2, \quad l \le -1. \tag{3.41}$$

Proof. The first equation follows from the first identity in (3.4), upon making the substitution $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$. The second equation is verified easily. Note that $b_l > 0$ for every integer l and $\sum_{l\in\mathbf{Z}} b_l < \infty$. ∎

Corollary 3.10. *Let $M \ge 2$ be an integer and let $x_j = \exp(i2\pi j/M)$, $0 \le j \le M-1$. Let $c = (c_0, \ldots, c_{M-1}) \in \mathbf{R}^M$ and define*

$$\hat{c}_r := \sum_{j=0}^{M-1} c_j e^{\frac{i2\pi jr}{M}}, \qquad r \in \mathbf{Z}. \tag{3.42}$$

Then

$$\sum_{j,k=0}^{M-1} c_j c_k g(d_1(x_j, x_k)) = \sum_{k=0}^{M-1} \left[\sum_{r\in\mathbf{Z}} b_{rM+k}\right] |\hat{c}_k|^2. \tag{3.43}$$

Proof. Lemma 3.9 and the obvious equations $\hat{c}_{l+M} = \hat{c}_l$, $l \in \mathbf{Z}$, provide the following:

$$\begin{aligned}\sum_{j,k=0}^{M-1} c_j c_k g(d_1(x_j, x_k)) &= \sum_{l\in\mathbf{Z}} b_l |\hat{c}_l|^2 = \sum_{r\in\mathbf{Z}} \sum_{k=0}^{M-1} b_{rM+k} |\hat{c}_{rM+k}|^2 \\ &= \sum_{r\in\mathbf{Z}} \sum_{k=0}^{M-1} b_{rM+k} |\hat{c}_k|^2 = \sum_{k=0}^{M-1} \left[\sum_{r\in\mathbf{Z}} b_{rM+k}\right] |\hat{c}_k|^2.\end{aligned} \tag{3.44}$$

∎

Theorem 3.11. *Let $M \ge 2$ be an integer and let $x_j = \exp(i2\pi j/M)$, $0 \le j \le M-1$. Suppose g, $\{a_l\}$ and $\{b_l\}$ are as before and define*

$$B := \min_{0\le k\le M-1} \sum_{r\in\mathbf{Z}} b_{rM+k} \quad \text{and} \quad \hat{B} := \max_{0\le k\le M-1} \sum_{r\in\mathbf{Z}} b_{rM+k}. \tag{3.45}$$

The following hold:

(i)

$$\min\left\{\sum_{j,k=0} c_j c_k g(d_1(x_j, x_k)) \,:\, c = (c_0, \ldots, c_{M-1}) \in \mathbf{R}^M,\ \|c\|_2 = 1\right\} = BM \tag{3.46}$$

and

$$\max\left\{\sum_{j,k=0} c_j c_k g(d_1(x_j,x_k)) \,:\, c=(c_0,\ldots,c_{M-1})\in\mathbf{R}^M,\ \|c\|_2=1\right\}=\hat{B}M. \tag{3.47}$$

(ii) If $A_M := (g(d_1(x_j,x_k)))_{j,k=0}^{M-1}$*, then*

$$cond_2(A_M)=\frac{\hat{B}}{B}. \tag{3.48}$$

Proof. (i) Define $\hat{c}=(\hat{c}_0,\ldots,\hat{c}_{M-1})$, where $\hat{c}_r$ is given by equation (3.42). Let Q be the $M\times M$ orthogonal matrix defined by

$$Q_{jk} := M^{-1/2}\exp(i2\pi jk/M), \qquad 0\le j,k\le M-1. \tag{3.49}$$

Then

$$\hat{c}=\sqrt{M}Qc \quad \text{and hence} \quad \|\hat{c}\|_2^2 = M\,\|c\|_2^2. \tag{3.50}$$

Combining the second equation above with Corollary 3.10, we see that

$$\min\left\{\sum_{j,k=0} c_j c_k g(d_1(x_j,x_k)) \,:\, c=(c_0,\ldots,c_{M-1})\in\mathbf{R}^M,\ \|c\|_2=1\right\}\ge BM \tag{3.51}$$

and

$$\max\left\{\sum_{j,k=0} c_j c_k g(d_1(x_j,x_k)) \,:\, c=(c_0,\ldots,c_{M-1})\in\mathbf{R}^M,\ \|c\|_2=1\right\}\le \hat{B}M. \tag{3.52}$$

To see that equality holds in (3.51), let us define $\hat{c}^{(0)}=(\hat{c}_0^{(0)},\ldots,\hat{c}_{M-1}^{(0)})\in\mathbf{R}^M$ by

$$\hat{c}_k^{(0)}=\begin{cases}1, & \text{if } k=k_0;\\ 0, & \text{otherwise,}\end{cases} \tag{3.53}$$

where

$$B=\sum_{r\in\mathbf{Z}} b_{rM+k_0}. \tag{3.54}$$

Let $c^{(0)}=(c_0^{(0)},\ldots,c_{M-1}^{(0)})\in\mathbf{R}^M$ be defined by

$$c^{(0)}=M^{-1/2}Q^*\hat{c}^{(0)}, \tag{3.55}$$

so that

$$\hat{c}^{(0)}=\sqrt{M}Qc^{(0)} \quad \text{and} \quad \|c^{(0)}\|_2 = M^{-1/2}. \tag{3.56}$$

We see from (3.43) that equality holds in (3.51) for the specific choice $c = \sqrt{M} c^{(0)}$. A similar argument shows that that equality also holds in (3.52), thereby finishing the proof of (i).

(ii) We see from (i) that

$$\|A_M\|_2 = \hat{B} M \quad \text{and} \quad \|A_M^{-1}\|_2 = (BM)^{-1}. \tag{3.57}$$

■

A simple and usable corollary of the preceding result is given below; its (straightforward) proof is omitted.

Corollary 3.12. *Suppose M, g, $\{a_l\}$, $\{b_l\}$, and $(x_j)_0^{M-1}$ are as in Theorem 3.11. If $b_\nu \geq b_{\nu+1}$ for every nonnegative integer ν, then*

$$B = \sum_{r=1}^{\infty} b_{rM-1} + \sum_{r=1}^{\infty} b_{rM} \quad \textit{and} \quad \hat{B} = b_0 + b_1 + \sum_{r=1}^{\infty} b_{rM} + \sum_{r=1}^{\infty} b_{rM+1}. \tag{3.58}$$

■

Example 3.13. Suppose $g(u) = \sum_{l=0}^{\infty} a_l \cos(lu)$, $u \in [-\pi, \pi]$, where $a_0 = 1$ and $a_l = l^{-\alpha}$, $l \geq 1$, for some fixed number α greater than 1. Suppose $M \geq 2$ is an integer and let $x_j = \exp(i2\pi j/M)$, $0 \leq j \leq M-1$. If $A_M := (g(d_1(x_j, x_k)))_{j,k=0}^{M-1}$, then

$$\|A_M\|_2 \asymp M, \quad \|A_M^{-1}\|_2 \asymp M^{\alpha-1}, \qquad \text{hence} \qquad \text{cond}_2(A_M) \asymp M^\alpha. \tag{3.59}$$

Discussion. Apply Corollary 3.12 and Theorem 3.11. ■

Remark 3.14. The DFT methods used above may be combined with [DNW, Appendix] to derive estimates for matrices involving 'Kadec-type' interpolation-points on the unit circle, that is, points x_j of the form

$$x_j = e^{i\phi_j}, \qquad \phi_j = (2\pi/M)\,(j + \epsilon_j), \qquad |\epsilon_j| \leq L < 1/4, \qquad j = 0, \ldots, M-1. \tag{3.60}$$

However, we shall refrain from elaborating on this here.

Acknowledgment

It is a pleasure to thank Joachim Stöckler for his comments. The work of F. J. Narcowich was supported by the Air Force Office of Scientific Research through Grant F49620-92-J-0403DEF, and that of J. D. Ward was supported by AFOSR Grant F49620-92-J-0403DEF and by the National Science Foundation through Grant No: DMS-9206928. N. Sivakumar is supported by NSF Grant No: DMS-9706583.

Note added in revised version

Subsequent to the completion of this work, we received the two preprints [LLS] and [M3], both of which deal with the same subject matter addressed in our paper. The approaches taken in these two articles are different from one another, and both of these approaches are different from that of ours.

References

[B] Ball, K., Eigenvalues of Euclidean distance matrices, *J. Approx. Theory*, **68** (1992), 74–82.

[Ba] Baxter, B. J. C., Norm estimates for inverses of Toeplitz distance matrices, *J. Approx. Theory*, **79** (1994), 222–242.

[Bu] Buhmann, M. D., New developments in the theory of radial basis function interpolation, in *Multivariate approximation: From CAGD to Wavelets*, K. Jetter and F. I. Utreras (eds.), World Scientific Publishing Company, Singapore, 1993, 35–76.

[BSW] Ball, K., N. Sivakumar and J. D. Ward, On the sensitivity of radial basis interpolation to minimal data separation distance, *Constr. Approx.,* **8** (1992), 401–426.

[C] Cheney, E. W., Approximation using positive definite functions, in *Approximation Theory VIII, Vol 1: Approximation and Interpolation*, C. K. Chui and L. L. Schumaker (eds.), World Scientific Publishing Company, Singapore, 1996, 145–168.

[D] Dyn, N., Interpolation and approximation by radial and related functions, in *Approximation Theory VI, Vol. 1*, C. K. Chui, L. L. Schumaker and J. D. Ward (eds.), Academic Press, New York, 1989, 211–234.

[DNW] Dyn, N., F. J. Narcowich and J. D. Ward, Variational principles and Sobolev-type estimates for generalized interpolation on a Riemannian manifold, *Constr. Approx.,* to appear.

[F1] Fasshauer, G. E., Hermite interpolation with radial basis functions on spheres, preprint.

[F2] Fasshauer, G. E., On the density of certain classes of radial basis functions on spheres, preprint.

[F3] Fasshauer, G. E., Scattered data interpolation with radial basis functions on the sphere, preprint.

[I] Ingham, A. E., Some trigonometric inequalities with applications to the theory of series, *Math. Z.,* **41** (1936), 367–379.

[JS] Jetter, K. and J. Stöckler, A generalization of de Boor's stability result and symmetric preconditioning, *Adv. Comput. Math.,* **3** (1995), 353–367.

[LLS] Levesley, J., Z. Luo and X. Sun, Norm estimates of interpolation matrices and their inverses associated with strictly positive definite functions, preprint, 1997.

[M1] Menegatto, V. A., Interpolation on spherical domains, *Analysis*, **14** (1994), 415–424.

[M2] Menegatto, V. A., Strictly positive definite functions on the circle, *Rocky Mountain J. Math.,* **25** (1995), 1149–1163.

[M3] Menegatto, V. A., Condition numbers associated with radial-function interpolation on spheres, preprint, 1997.

[Mi] Micchelli, C. A., Interpolation of scattered data: distance matrices and conditionally positive definite functions, *Constr. Approx.,* **2** (1986), 11–22.

[Mü] Müller, C., *Spherical Harmonics*, Springer-Verlag, Berlin, 1966.

[MN1] Madych W. R. and S. A. Nelson, Multivariate interpolation and conditionally positive definite functions, I, *Approx. Theory Appl.,* **4** (1988), 77–89.

[MN2] Madych W. R. and S. A. Nelson, Multivariate interpolation and conditionally positive definite functions, II, *Math. Comp.,* **54** (1990), 211–230.

[N] Narcowich, F. J., Generalized Hermite interpolation and positive definite kernels on a Riemannian manifold, *J. Math. Anal. Appl.,* **190** (1995), 165–193.

[NW1] Narcowich, F. J. and J. D. Ward, Norms of inverses and condition numbers for matrices associated with scattered data, *J. Approx. Theory*, **64** (1991), 69–94.

[NW2] Narcowich, F. J. and J. D. Ward, Norm estimates for the inverses of a general class of scattered-data radial-function interpolation matrices, *J. Approx. Theory*, **69** (1992), 84–109.

[NW3] Narcowich, F. J. and J. D. Ward, Nonstationary wavelets on the m-sphere for scattered data, *Appl. Comput. Harmonic Anal.,* **3** (1996), 1–13.

[NSW] Narcowich, F. J., N. Sivakumar and J. D. Ward, On condition numbers associated with radial-function interpolation, *J. Math. Anal. Appl.,* **186** (1994), 457–485.

[P] Powell, M. J. D., The theory of radial basis function approximation in 1990, in *Advances in Numerical Analysis II*, W. A. Light (ed.), Clarendon Press, Oxford, 1992, 106–210.

[RS] Ron, A. and X. Sun, Strictly positive definite functions on spheres in Euclidean spaces, *Math. Comp.,* **65** (1996), 1513–1530.

[S] Schaback, R., Lower bounds for norms of inverses of interpolation matrices for radial basis functions, *J. Approx. Theory*, **79** (1994), 287–306.

[Sc1] Schoenberg, I. J., Positive definite functions on spheres, *Duke Math. J.,* **9** (1942), 96–108.

[Sc2] Schoenberg, I. J., Contributions to the problem of approximation of equidistant data by analytic functions, Part A, *Quart. Appl. Math.,* **IV** (1946), 45–99.

[Sc3] Schoenberg, I. J., Cardinal interpolation and spline functions, *J. Approx. Theory*, **2** (1969), 167–206.

[Sch] Schreiner, M., On a new condition for strictly positive definite functions on spheres, *Proc. Amer. Math. Soc.,* **125** (1997), 531–539.

[Su] Sun, X., Norm estimates for inverses of Euclidean distance matrices, *J. Approx. Theory*, **70** (1992), 339–347.

[XC] Xu, Y. and E. W. Cheney, Strictly positive definite functions on spheres, *Proc. Amer. Math. Soc.,* **116** (1992), 977–981.