

## SURFACES

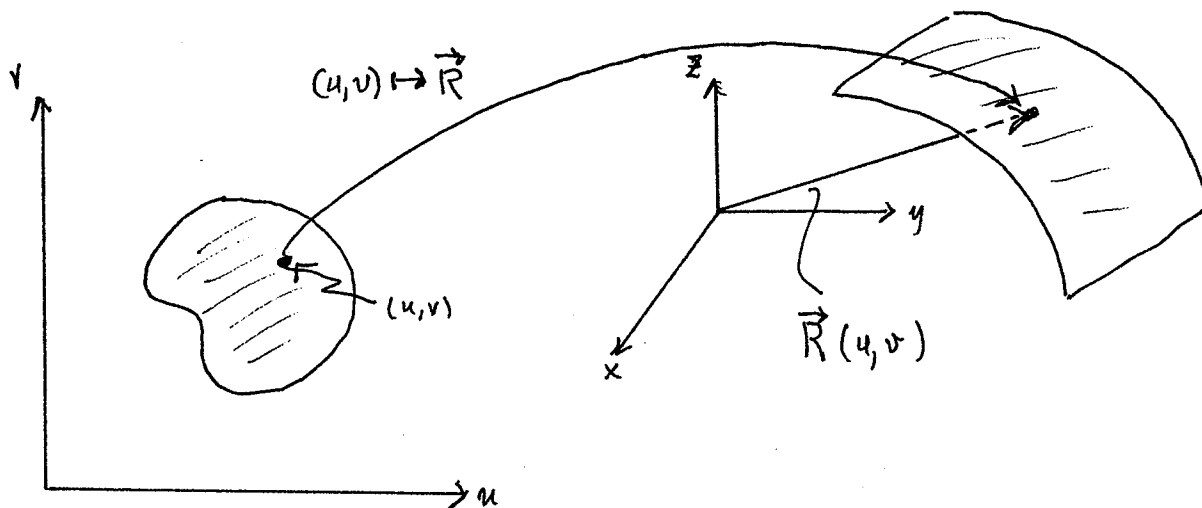
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**I. Surfaces.** We will think of a surface as a set of points  $\{(x, y, z)\}$  in  $\mathbb{R}^3$  that is either *implicitly* given by an equation of the form  $F(x, y, z) = 0$ , or is *parametrically* given in the (vector) form

$$\mathbf{R}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k},$$

where  $(u, v)$  belongs to some set of points in  $\mathbb{R}^2$ . The variables  $u$  and  $v$  are called *parameters*.



**Fig. 1.** Parametric representation of surface.

Many surfaces can be represented in both ways. A *sphere* with radius  $a$  and center  $(0, 0, 0)$  is implicitly given by  $x^2 + y^2 + z^2 = a^2$ , and is parametrically represented by

$$\mathbf{R}(\theta, \phi) = a \sin \phi \cos \theta \mathbf{i} + a \sin \phi \sin \theta \mathbf{j} + a \cos \phi \mathbf{k},$$

where  $0 \leq \theta \leq 2\pi$  and  $0 \leq \phi \leq \pi$ . These are the usual spherical coordinates with  $\rho = a$ . A *cylinder* with radius  $a$  and central axis coinciding with the  $z$ -axis has the explicit representation  $x^2 + y^2 = a^2$ , and is implicitly given by

$$\mathbf{R}(\theta, z) = a \cos \theta \mathbf{i} + a \sin \theta \mathbf{j} + z \mathbf{k},$$

where  $0 \leq \theta \leq 2\pi$  and  $z$  depends on the height. To get this parametrization, we used cylindrical coordinates with  $r = a$ .

Planes may also be represented both ways. If  $\mathbf{N} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$  is a vector normal to a plane  $\mathcal{P}$ , and if  $(x_0, y_0, z_0)$  is a point on  $\mathcal{P}$ , then the implicit equation for it is  $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ . In vector form, this equation is just  $\mathbf{N} \cdot (\mathbf{R} - \mathbf{R}_0) = 0$ . To get a parametric representation for  $\mathcal{P}$ , we need two non-colinear vectors  $\mathbf{p}$  and  $\mathbf{q}$  that are parallel to  $\mathcal{P}$ —or what is the same, are perpendicular to  $\mathbf{N}$ . (See Fig. 2). Looking at the diagram, it is easy to see that the (vector) parametric representation for a plane is

$$\mathbf{R}(u, v) = \mathbf{R}_0 + u\mathbf{p} + v\mathbf{q}.$$

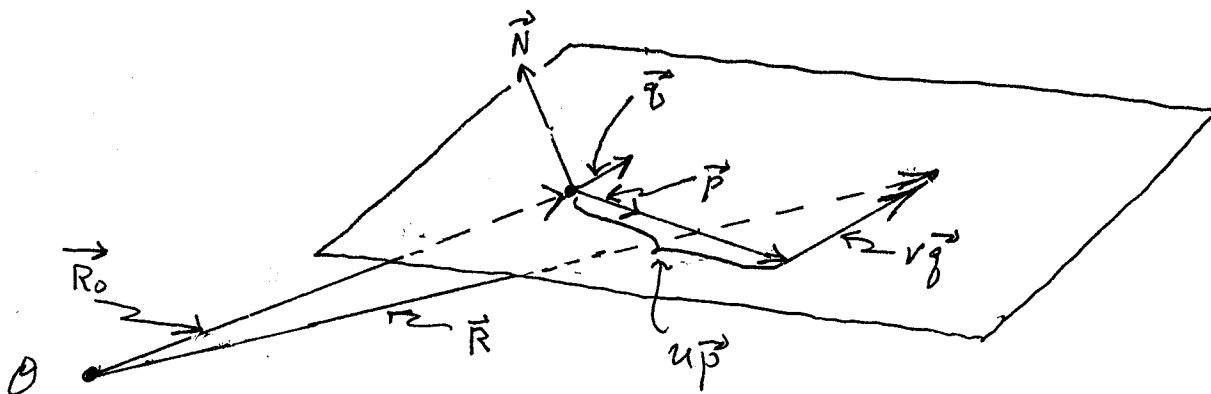


Fig. 2. Parametric representation of a plane.

Our final example of a parametric surface is the *torus*. Geometrically, we obtain a torus by revolving a circle in the  $x$ - $y$  plane about the  $z$ -axis. Thus, in every  $\theta = \text{const.}$  plane the cross section of the torus is the circle shown below in Fig. 3. A little geometry then gives us that  $z = b \sin \psi$  and  $r = a + b \cos \psi$ . Using the relationship between cartesian and cylindrical coordinates then yields

$$\mathbf{R}(\theta, \psi) = (a + b \cos \psi) \cos \theta \mathbf{i} + (a + b \cos \psi) \sin \theta \mathbf{j} + b \sin \psi \mathbf{k}$$

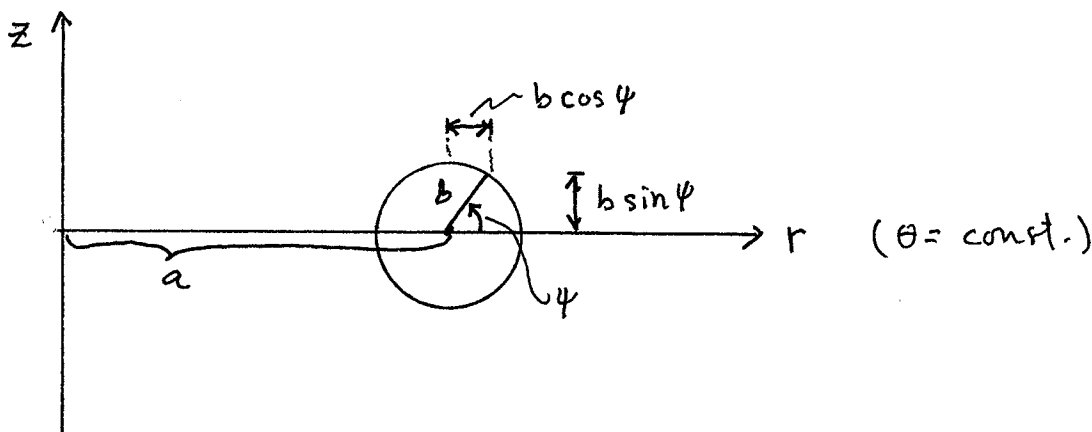


Fig. 3. Cross section of a torus.

**II. Elements of surface area.** In dealing with surface integrals, whether we are working with a “density” integral or a “flux” integral, we are really doing a “sum” in which we are adding multiples of the area element for the surface we are integrating over. The idea is that we make a fine mesh in the parameter space, and use this to create a mesh on our surface. This is illustrated in Fig. 4 below.

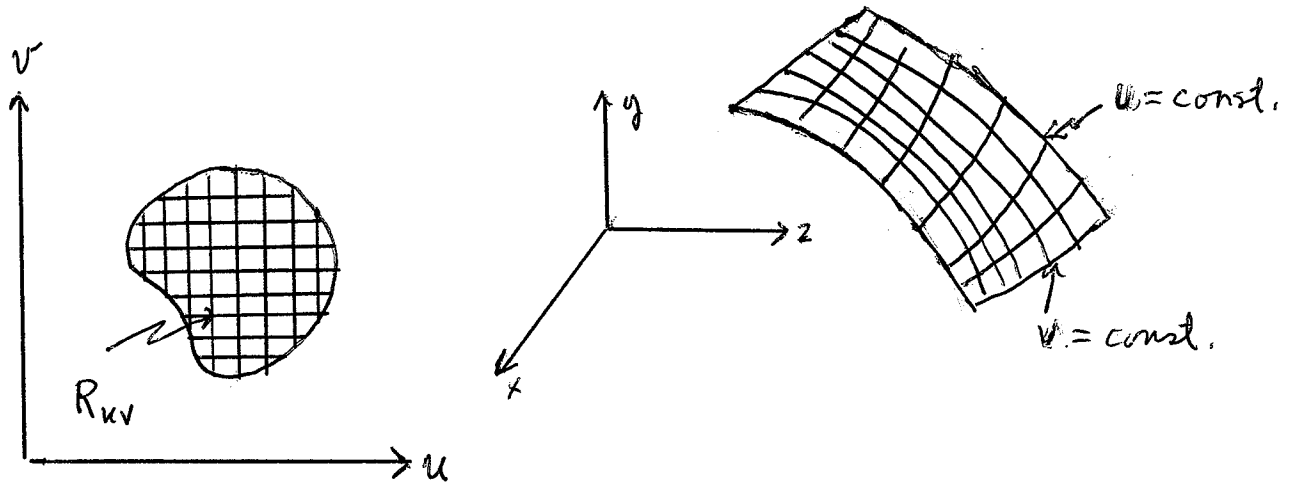


Fig. 4. Mesh for approximating a surface integral.

Within a given mesh parallelogram on the surface, the function that we wish to integrate is approximately constant. The area,  $\Delta\sigma$ , of the mesh parallelogram (element of surface area—see Fig. 5) specified by the point  $(u, v)$  in parameter space is approximately

$$\Delta\sigma \approx \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| \Delta u \Delta v.$$

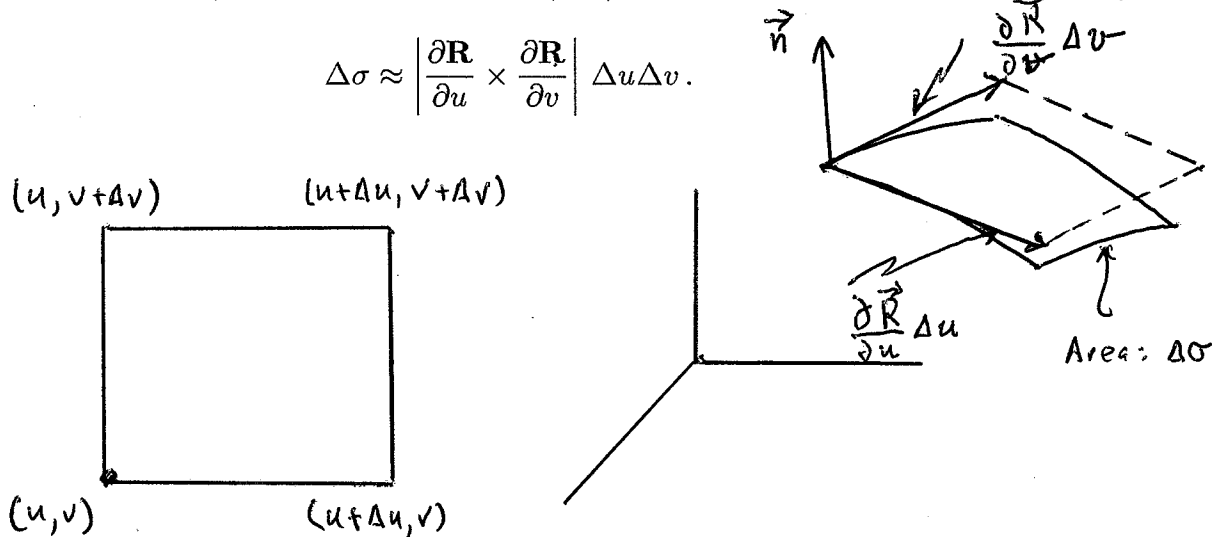


Fig. 5. Element of surface area.

The surface integrals for a function  $g(\mathbf{R})$  and for a flux density  $\mathbf{F}(\mathbf{R}) \cdot \mathbf{n}$  are approximated by the sums

$$\sum_{mesh} g(\mathbf{R}) \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| \Delta u \Delta v \quad \text{and} \quad \sum_{mesh} \mathbf{F}(\mathbf{R}) \cdot \mathbf{n} \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| \Delta u \Delta v,$$

respectively. (Here,  $\mathbf{n}$  is the appropriate unit normal to the surface  $S$ ; see Fig. 5.) In the limit, we get the following double integral formulas for the two types of surface integrals:

$$\begin{aligned} \iint_S g(\mathbf{R}) d\sigma &= \iint_{R_{uv}} g(\mathbf{R}(u, v)) \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| du dv \\ \iint_S \mathbf{F}(\mathbf{R}) \cdot \mathbf{n} d\sigma &= \iint_{R_{uv}} \mathbf{F}(\mathbf{R}(u, v)) \cdot \mathbf{n}(u, v) \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| du dv. \end{aligned}$$

Probably the easiest way to remember these formulas is to think of expressing the element of surface area  $d\sigma$  and normal vector  $\mathbf{n}$  in parametric form. The element of surface area is

$$d\sigma = \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| du dv.$$

Since the vectors  $\frac{\partial \mathbf{R}}{\partial u}$  and  $\frac{\partial \mathbf{R}}{\partial v}$  are in the plane tangent to the surface at  $\mathbf{R}(u, v)$ , the vector

$$\mathbf{N} = \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v}$$

is normal to the surface; indeed,  $\mathbf{N}$  is called the *standard normal*. This vector is parallel to the *unit normal*  $\mathbf{n}$ , which we can express as  $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ . Note that the expression for the surface area element can also be given in terms of  $\mathbf{N}$ :

$$d\sigma = |\mathbf{N}| du dv.$$

In a flux integral, we can use this formula as follows:

$$\begin{aligned} \iint_S \mathbf{F}(\mathbf{R}) \cdot \mathbf{n} d\sigma &= \iint_{R_{uv}} \mathbf{F}(\mathbf{R}(u, v)) \cdot \mathbf{n}(u, v) \left| \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} \right| du dv \\ &= \iint_{R_{uv}} \mathbf{F}(\mathbf{R}(u, v)) \cdot \frac{\mathbf{N}(u, v)}{|\mathbf{N}|} |\mathbf{N}| du dv \\ &= \iint_{R_{uv}} \mathbf{F}(\mathbf{R}(u, v)) \cdot \mathbf{N}(u, v) du dv. \end{aligned}$$

This formula is useful because it eliminates the need for computing the length of the standard normal vector. In the next section we will give examples of standard normals and surface area elements in a few important cases.

**III. Examples.** We will close by giving the surface area element and standard normal for the examples discussed in the first section.

**The sphere with center (0,0) and radius  $a$ .**

$$\text{Standard (outward) normal: } \mathbf{N} = \frac{\partial \mathbf{R}}{\partial \phi} \times \frac{\partial \mathbf{R}}{\partial \theta} = a \sin \phi \mathbf{R}(\theta, \phi)$$

$$\text{Element of surface area: } d\sigma = a^2 \sin \phi d\theta d\phi.$$

**The cylinder with central axis  $z$  and radius  $a$ .**

$$\text{Standard (outward) normal: } \mathbf{N} = \frac{\partial \mathbf{R}}{\partial \phi} \times \frac{\partial \mathbf{R}}{\partial z} = a^2 (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$$

$$\text{Element of surface area: } d\sigma = a d\theta dz.$$

**The plane  $\mathbf{R}(u, v) = \mathbf{R}_0 + u\mathbf{p} + v\mathbf{q}$ .**

$$\text{Standard normal: } \mathbf{N} = \frac{\partial \mathbf{R}}{\partial u} \times \frac{\partial \mathbf{R}}{\partial v} = \mathbf{p} \times \mathbf{q}$$

$$\text{Element of surface area: } d\sigma = |\mathbf{p} \times \mathbf{q}| du dv.$$

**The torus obtained by revolving  $(x - a)^2 + y^2 = b^2$  about the  $z$ -axis.**

$$\begin{aligned} \text{Standard (outward) normal: } \mathbf{N} = \frac{\partial \mathbf{R}}{\partial \theta} \times \frac{\partial \mathbf{R}}{\partial \psi} = & (a + b \cos \psi)(b \cos \psi \cos \theta \mathbf{i} \\ & + b \cos \psi \sin \theta \mathbf{j} + b \sin \psi \mathbf{k}) \end{aligned}$$

$$\text{Element of surface area: } d\sigma = b(a + b \cos \psi) d\theta d\psi.$$