Notes on Surfaces

Francis J. Narcowich
Department of Mathematics
Texas A&M University

I. Surfaces. We will think of a surface as a set of points $\{(x, y, z)\}$ in $\mathbb{R}^3$ that is either *implicitly* given by an equation of the form $F(x, y, z) = 0$, or is *parametrically* given in the (vector) form

$$\mathbf{x}(s,t) = x(s,t)\,\mathbf{i} + y(s,t)\,\mathbf{j} + z(s,t)\,\mathbf{k},$$

where (s, t) belongs to some region D in $\mathbb{R}^2$. The variables s and t are called *parameters.*

Many surfaces can be represented in both ways. A *sphere* with radius a and center $(0, 0, 0)$ is implicitly given by $x^2 + y^2 + z^2 = a^2$, and is parametrically repesented by

$$\mathbf{x}(\theta,\,\phi) = a\sin\phi\cos\theta\,\mathbf{i} + a\sin\phi\sin\theta\,\mathbf{j} + a\cos\phi\,\mathbf{k}\,,$$

where $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$, which are the paramters, are the usual spherical coördinates with $\rho = a$. A *cylinder* with radius a and central axis coinciding with the z-axis has the explicit representation $x^2 + y^2 = a^2$, and is implicitly given by

$$\mathbf{x}(\theta, z) = a\cos\theta\,\mathbf{i} + a\sin\theta\,\mathbf{j} + z\,\mathbf{k}\,,$$

where $0 \le \theta \le 2\pi$ and z are the parameters. To get this parametrization, we used cylindrical coördinates with $r = a$.

Planes may also be represented both ways. Since the parametric representation is less familiar, we discuss it. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-collinear vectors in a plane $\mathcal{P}$ and that $\mathbf{x}_0 = x_0\,\mathbf{i} + y_0\,\mathbf{j} + z_0\,\mathbf{k}$ is a point on $\mathcal{P}$. To get the (vector) parametric representation for a plane, first go from the origin to a point $\mathbf{x}_0$ on the plane, and then use the vectors $\mathbf{u}$, $\mathbf{v}$, which are a basis for vectors parallel to the plane, to write $\mathbf{x}(s,t) - \mathbf{x}_0 = s\,\mathbf{u} + t\,\mathbf{v}$. Solving, we get

$$\mathbf{x}(s,t) = \mathbf{x}_0 + s\,\mathbf{u} + t\,\mathbf{v}.$$

II. Elements of surface area. In dealing with surface integrals, whether we are working with a scalar integral or a vector integral, we are really doing a "sum" in which we are adding multiples of the area element for the surface we are integrating over. The idea is that we make a fine mesh in the parameter space, which creates a curved mesh on the surface. The rectangles in the $s - t$ mesh become (approximate) parallelograms on the surface, with sides $(\partial\,\mathbf{x}/\partial s)\Delta s$ and $(\partial\,\mathbf{x}/\partial t)\Delta t$. Within a given mesh parallelogram on the surface, the function that we wish to integrate is approximately constant. The area, ΔS, of the mesh parallelogram (element of surface area) specified by the point (s, t) in parameter space is approximately

$$\Delta S \approx \|\mathbf{N}\|\Delta s\Delta t\,,\ \text{where } \mathbf{N} := \frac{\partial\,\mathbf{x}}{\partial s} \times \frac{\partial\,\mathbf{x}}{\partial t}.$$

Since the vectors $\frac{\partial \mathbf{x}}{\partial s}$ and $\frac{\partial \mathbf{x}}{\partial t}$ are in the plane tangent to the surface at $\mathbf{x}(s,t)$, the vector $\mathbf{N}$ is normal (perpendicular) to the surface; it is called the *standard normal.* The vector

$$\mathbf{n} := \mathbf{N}/\|\mathbf{N}\|,$$

which has length 1 and is also normal to S, is called the *unit* normal. The surface integrals for a function $f(\mathbf{x})$ and for a vector field $\mathbf{F}(\mathbf{x})$ are approximated by the sums

$$\sum_{mesh} f(\mathbf{x})\|\mathbf{N}\|\,\Delta s\Delta t \quad \text{and} \quad \sum_{mesh} \mathbf{F}(\mathbf{x})\cdot\mathbf{n}\left\|\frac{\partial \mathbf{x}}{\partial s}\times\frac{\partial \mathbf{x}}{\partial t}\right\|\,\Delta s\Delta t\,,$$

respectively. In the limit, we get the following double integral formulas for scalar and vector surface integrals:

$$\iint_S f(\mathbf{x})dS = \iint_D f(\mathbf{x}(s,t))\|\mathbf{N}(\mathbf{x}(s,t))\|\,dsdt$$

$$\iint_S \mathbf{F}(\mathbf{x})\cdot\mathbf{n}dS = \iint_D \mathbf{F}\cdot\mathbf{n}|\mathbf{N}|\,dsdt = \iint_D \mathbf{F}(\mathbf{x}(s,t))\cdot\mathbf{N}(\mathbf{x}(s,t))\,dsdt\,.$$

This formula for the vector integral is useful because it eliminates the need for computing the length of the standard normal vector. In the next section we will give examples of standard normals and surface area elements in a few important cases.

III. Examples. We will close by giving the surface area element and standard normal for the examples discussed in the first section.

The sphere with center (0,0) and radius a.

$$\text{Standard (outward) normal: } \mathbf{N} = \frac{\partial \mathbf{x}}{\partial \theta}\times\frac{\partial \mathbf{x}}{\partial \phi} = a\sin\phi\,\mathbf{x}(\theta,\phi)$$

$$\text{Element of surface area: } dS = a^2\sin\phi\,d\theta\phi\,.$$

The cylinder with central axis z and radius a.

$$\text{Standard (outward) normal: } \mathbf{N} = \frac{\partial \mathbf{x}}{\partial \phi}\times\frac{\partial \mathbf{x}}{\partial z} = a^2\,(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$$

$$\text{Element of surface area: } dS = a\,d\theta dz\,.$$

The plane $\mathbf{x}(x,t) = \mathbf{x}_0 + s\mathbf{u} + t\mathbf{v}$.

$$\text{Standard normal: } \mathbf{N} = \frac{\partial \mathbf{x}}{\partial s}\times\frac{\partial \mathbf{x}}{\partial t} = \mathbf{u}\times\mathbf{v}$$

$$\text{Element of surface area: } dS = |\mathbf{u}\times\mathbf{v}|\,dsdt\,.$$

The torus obtained by revolving $(x-a)^2 + y^2 = b^2$ about the z-axis.

$$\mathbf{x}(\theta,\,\psi) = (a + b\cos\psi)(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}) + b\sin\psi.$$

$$\text{Standard (outward) normal: } \mathbf{N} = \frac{\partial \mathbf{x}}{\partial \theta}\times\frac{\partial \mathbf{x}}{\partial \psi} = (a + b\cos\psi)(b\cos\psi\cos\theta\,\mathbf{i} + b\cos\psi\sin\theta\,\mathbf{j} + b\sin\psi\,\mathbf{k})$$

$$\text{Element of surface area: } dS = b(a + b\cos\psi)d\theta d\psi\,.$$