APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 12, 2018

Applied Analysis Part, 2 hours

Name: ____

Instructions: *Do problemss 1* **and** *2 and either 3* **or** *4.* **No** extra credit for doing 3 and 4.

Problem 1. Let $\mathcal{D}$ be the set of compactly supported functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.

(b) Show that $\psi \in \mathcal{D}$ satisfies $\psi = \phi''$ for some $\phi \in \mathcal{D}$ if and only if

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \text{ and } \int_{-\infty}^{\infty} x\psi(x)dx = 0.$$

(c) Find all distributions $T \in \mathcal{D}'$ such that $T''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1)$.

Problem 2. Consider a functional $K[u]$, where $u \in V$, and V is a Banach space.

a Define the Frechét derivative and the Gâteaux derivative for $K[u]$. Use a simple two-dimensional example to illustrate the difference between the two types of derivatives.

b Let $p(x) \in C^2[0,1]$, $p(x) \geq c > 0$. Consider the constrained functional,

$$J[u] = \int_0^1 pu'^2 dx + \sigma u(1)^2, \; H[u] = \int_0^1 u^2 dx = 1,$$

where $u \in C^{(1)}[0,1]$, $u(0) = 0$, and $\sigma > 0$. Calculate the variational derivative of the problem, using Lagrange multipliers. Find the Sturm-Liouville eigenvalue problem associated with it.

c How does the second eigenvalue of this problem compare with the second eigenvalue of the corresponding Dirichlet problem, i.e., $u(0) = u(1) = 0$? with the mixed Dirichlet-Neumann problem $u(0) = 0 = u'(1)$? Prove your answer.

Problem 3. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u(0) = 0$; that is, L has the domain

$$D_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u(0) = 0\}.$$

Find the Green's function G satisfying $-G'' - zG = \delta(x-\xi)$, with $G(0,\xi;z) = 0$, where $z \in \mathbb{C} \setminus [0,\infty)$.

Problem 4. Consider the kernel $k(x,y) = \sum_{n=0}^{\infty}(1+n)^{-4}P_{n+1}(x)P_n(y)$, where P_n is the n^{th} Legendre polynomial, normalized so that $\int_{-1}^{1} P_n(x)^2 dx = \frac{2}{2n+1}$.

(a) Show that $Ku(x) = \int_{-1}^{1} k(x,y)u(y)dy$ is a compact operator on $L^2[-1,1]$.

(b) Determine the spectrum of K.

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January 12, 2018

Numerical Analysis Part, 2 hours

Name: ______________________________

Instructions: *Do all problems 1-3 in this part of the exam; problem 4 is a bonus question. Show all of your work clearly.*

Problem 1. Let K be a triangle in $\mathbb{R}^2$. Denote by $|K|$ the area of K. Let m_1, m_2, and m_3 be the mid-points of the three edges. Here $H^m(\Omega)$ is the standard Sobolev space of functions defined on Ω that have square integrable weak derivatives of order m and $\mathcal{P}_k$ is the set of polynomials of degree k.

(a) Prove that the following quadrature is exact for every polynomial in $\mathcal{P}_2$:

$$\int_K p(x)dx = \frac{1}{3}|K|(p(m_1) + p(m_2) + p(m_3)).$$

(b) Let h_K be the diameter of K. Prove that there is $c > 0$ (depending on the triangle K) s.t.

$$\forall v \in H^3(K), \qquad \left|\int_K v(x)dx - \frac{1}{3}|K|(v(m_1) + v(m_2) + v(m_3))\right| \le ch_K^3|K|^{\frac{1}{2}}|v|_{H^3(K)}.$$

Note: You may use the Bramble-Hilbert Lemma without proof as long as you state it correctly before using it.

Problem 2. Let V be a closed subspace of $H^1(\Omega)$, $V_h \subset V$ be a finite element approximation space and Ω a domain in $\mathbb{R}^d$. We consider the Crank-Nicolson approximation in time: find $W^j \in V_h$, $j = 0, 1, \ldots$ satisfying

$$\Big(\frac{W^{n+1} - W^n}{k}, \theta\Big) + \frac{1}{2}A(W^{n+1} + W^n, \theta) = (f^{n+\frac{1}{2}}, \theta), \qquad \forall \theta \in V_h.$$

Here $k > 0$ is the time step size, $t_n = nk$, $f^{n+\frac{1}{2}}(\cdot) = f(\cdot, t_n + \frac{k}{2}) \in V_h$, $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and $A(\cdot,\cdot)$ is a symmetric, coercive, and bounded bilinear form on V.

Let $\{\psi_i\}$, $i = 1, \ldots, M$ be an orthonormal basis with respect to $(\cdot,\cdot)$ for V_h of eigenfunctions satisfying

$$A(\psi_i, \theta) = \lambda_i(\psi_i, \theta), \qquad \forall \theta \in V_h.$$

(a) Using the expansion

$$W^n = \sum_{i=1}^{M} c_i^n \psi_i \qquad \& \qquad f^{n+\frac{1}{2}} = \sum_{i=1}^{M} d_i^n \psi_i,$$

derive a recurrence relation for c_i^{n+1} in terms of $\delta_i = (1 - k\lambda_i/2)/(1 + k\lambda_i/2)$, c_i^n, k and d_i^n.

(b) Show that

$$|c_i^n| \le \begin{cases} |c_i^0| & \text{if} \quad f = 0, \\ \lambda_1^{-1/2} \left(k \sum_{j=0}^{n-1} |d_i^j|^2 \right)^{1/2} & \text{if} \quad W^0 = 0. \end{cases}$$

Here λ_1 is the smallest eigenvalue.

(c) Use Part (b) above and superposition principle to derive the stability estimate

$$\|W^n\| \le \|W^0\| + \lambda_1^{-1/2} \left(k \sum_{j=0}^{n-1} \|f^j\|^2 \right)^{1/2}.$$

Problem 3. Consider the boundary value problem: find $u(x)$ such that

$$\begin{aligned} -\Delta u + \alpha \frac{\partial u}{\partial x_1} + \beta x_1 \frac{\partial u}{\partial x_2} &= f(x), \quad x := (x_1, x_2) \in \Omega \\ u(x) &= 0, \qquad x \in \partial\Omega. \end{aligned} \tag{3.1}$$

Here Ω is a bounded convex polygonal domain in $\mathbb{R}^2$, α and β are given constants, and $f(x)$ is a given function in $L^2(\Omega)$. These guarantee full regularity of the solution for any α and β, i.e. $u \in H^2(\Omega)$ and $\|u\|_{H^2} \le C\|f\|_{L^2}$.

(a) Derive a weak form of this problem in an appropriate space V (identify this space !).
(b) Show that the corresponding bilinear form is coercive in the norm of the space V.
(c) Assume that you are given an admissible triangulation of the domain Ω and consider the space V_h of continuous piecewise linear functions with respect to this mesh vanishing on $\partial\Omega$. Assuming standard approximation properties of V_h, write down an *a priori* estimate for the error of the FEM in V-norm.
(d) Using the Aubin-Nitsche (duality) argument, derive an error estimate in the $L^2(\Omega)$-norm. Explain what additional regularity conditions are needed for this estimate.

Problem 4. (A bonus problem for extra 10 pts) Let K be a simplex in $\mathbb{R}^d$, $d > 1$ and let ρ_K be the diameter of the largest ball inscribed in K. Let ϕ_i, $i = 1, \dots, d+1$ be the nodal basis of the FE space of linear functions over K determined by their vertex values. Prove that

$$|\nabla \phi_i| \le \rho_K^{-1} \quad \text{for} \quad i = 1, \dots, d.$$