Applied/Numerical Analysis Qualifying Exam

August 13, 2014

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
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Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let f be a 2π-periodic function.

(a) Sketch a proof of the following: *If* f *is a piecewise* $C^{(1)}$ *(i.e., can have jumps), and if* $S_N = \sum_{n=-N}^{N} c_n e^{inx}$ *is the* N^{th} *partial sum of the Fourier series for* f*, then, for every* $x \in \mathbb{R}$,

$$\lim_{N\to\infty} S_N(x) = \frac{f(x^+) + f(x^-)}{2}.$$

(b) Show that if f is $C^{(1)}$, then the convergence is uniform.

Problem 2. Consider the boundary value problem

$$u'' = f, \; u(0) - u'(0) = 0, \; u(1) + u'(1) = 0. \tag{2.1}$$

(a) Find the Green's function, $G(x, y)$, for (2.1).

(b) Show that $Gf(x) = \int_0^1 G(x, y)f(y)dy$ is compact and self adjoint on $L^2[0, 1]$.

(c) State the spectral theorem for compact, self-adjoint operators. Use it to show that the (normalized) eigenfunctions of the eigenvalue problem $u'' + \lambda u = 0$, $u(0) - u'(0) = 0$, $u(1) + u'(1) = 0$ form a complete orthonormal set in $L^2[0, 1]$. (Hint: How are the eigenfunctions of G related to those of $u'' + \lambda u = 0$, $u(0) - u'(0) = 0$, $u(1) + u'(1) = 0$?)

Problem 3. Let $k(x, y) = x^2y^3$, $Ku(x) = \int_0^1 k(x, y)u(y)dy$, and $Lu = u - \lambda Ku$.

(a) Show that L has closed range.

(b) Determine the values of λ for which $Lu = f$ has a solution for all f. Solve $Lu = f$ for these values of λ.

(c) For the remaining values of λ, find a condition on f that guarantees a solution to $Lu = f$ exists. When f satisfies this condition, solve $Lu = f$.

Problem 4. Let $p \in C^{(2)}[0,1]$, and $q, w \in C[0,1]$, with $p, q, w > 0$. Consider the Sturm-Liouville (SL) eigenvalue problem, $(p\phi')' - q\phi + \lambda w\phi = 0$, subject to $\phi(0) = 0$ and either (A) $\phi(1) = 0$ or (B) $\phi'(1) + \phi(1) = 0$. In addition, for $\phi \in C^{(1)}[0,1]$, let $D[\phi] := \int_0^1 (p\phi'^2 + q\phi^2)dx$ and $H[\phi] := \int_0^1 w\phi^2 dx$.

(a) Show that minimizing the functional $D[\phi]$, subject to the constraint $H[\phi] = 1$ and boundary conditions $\phi(0) = \phi(1) = 0$, yields the SL problem (A).

(b) State the variational problem that will yield the SL problem (B). Verify that your answer is correct by calculating the variational (Fréchet) derivative and setting it equal to 0.

(c) State the Courant MINIMAX Principle. (Eigenvalues increase: $\lambda_1 < \lambda_2 < \lambda_3 \cdots$.) Use it to show that the n^{th} eigenvalue of the SL problem (A) is larger than or equal to the n^{th} eigenvalue of the SL problem (B).