Applied/Numerical Analysis Qualifying Exam

January 6, 2014

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
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Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let f be a continuous, 2π periodic function having the Fourier series $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}$. The trapezoidal rule for numerically finding $\int_0^{2\pi} f(t)dt$ is given by

$$Q_n(f) = \frac{2\pi}{n}\sum_{k=0}^{n-1} f(2\pi k/n).$$

(a) Let $S_m(t) = \sum_{k=-m}^{m} c_k e^{ikt}$. Show that $Q_n(S_{n-1}) = \int_0^{2\pi} f(t)dt$.
(b) Show that $\left|Q_n(f) - \int_0^{2\pi} f(t)dt\right| \le 2\pi \|f - S_{n-1}\|_{C[0,2\pi]}$.
(c) Suppose that $|c_k| \le |k|^{-6}$ for all $k \ne 0$. Estimate $\left|Q_n(f) - \int_0^{2\pi} f(t)dt\right|$.

Problem 2. Consider the Sturm-Liouville (S-L) problem

$$u'' = f,\ u'(0) = 0,\ u(1) + u'(1) = 0.$$

(a) Find the Green's function, $G(x, y)$, for this problem.
(b) Show that $Gf(x) = \int_0^1 G(x, y)f(y)dy$ is compact and self adjoint on $L^2[0, 1]$.
(c) Show that the eigenfunctions of the eigenvalue problem $u'' + \lambda u$, $u'(0) = 0$, $u(1) + u'(1) = 0$ form a complete set orthogonal set in $L^2[0, 1]$. (Hint: Show that the the null space of G is $\{0\}$.)

Problem 3. Find the first term of the asymptotic series for $F(x) := \int_0^\infty e^{xt - \frac{1}{2}t^2} dt,\ x \to +\infty$.

Problem 4. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. In addition, let the Fourier and inverse Fourier transforms be given by

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \text{ and } f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

(a) Define $\mathcal{S}$ and give the semi-norm topology for it. In addition, define $\mathcal{S}'$.
(b) Given that $\mathcal{F}$ is a continuous bijection mapping $\mathcal{S} \to \mathcal{S}$, define the Fourier transform of a tempered distribution.
(c) Show that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \ldots$.
(d) Let $T(t) = \begin{cases} 1 & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$. Show that $T'(t) = \delta(t+1) - \delta(t-1)$. Use (c) to find $\widehat{T}$.