APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 10, 2019

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any four problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all five.*

Problem 1. Let A be an $n \times n$ self-adjoint matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) State the Courant-Fischer mini-max theorem.
(b) Let $B = [b_1\ b_2]$ be a real $n \times 2$ matrix, with b_1, b_2 being linearly independent. Assume that $\|x\| = 1$. If $q(x) = x^T Ax$ and $\hat{q}(x) = q(x)|_{B^T x=0}$, show that

$$\lambda_3 \leq \max_{\|x\|=1} \hat{q}(x) \leq \lambda_1.$$

Problem 2. A sequence $\{f_n\}$ in $\mathcal{H}$ is said to be weakly convergent to $f \in \mathcal{H}$ if and only if $\lim_{n\to\infty}\langle f_n, g\rangle = \langle f, g\rangle$ for every $g \in \mathcal{H}$. When this happens, we write $f = w\text{-}\lim_{n\to\infty} f_n$. One can show that every weakly convergent sequence is bounded.

(a) Let $\{\phi_n\}_{n=1}^{\infty}$ be any orthonormal sequence. Show that $w\text{-}\lim_{n\to\infty} \phi_n = 0$. (Hint: use Bessel's inequality.)
(b) Let K be a compact linear operator on a Hilbert space $\mathcal{H}$. Show that if $w\text{-}\lim_{n\to\infty} f_n = f$, then $\lim_{n\to\infty} Kf_n = Kf$.
(c) Define $\rho(K)$, the resolvent set for K, and $\sigma(K)$, the spectrum of K. Use (a) and (b) to show that $0 \in \sigma(K)$.

Problem 3. Let $J[y] := \int_0^1 \left(\frac{1}{2}y'^2 + yy' + y' + y\right)dx$. Find the extremal of J that satisfies natural boundary conditions at $x = 0$ and $x = 1$.

Problem 4. Consider the operator $Lu = x^2u'' - xu'$ with domain $\mathcal{D}_L := \{u \in L^2[1,2] : Lu \in L^2[1,2],\ u(1) = 0\ \&\ u'(2) = 0\}$. You are given that the homogenous solutions of $Lu = 0$ are 1 and x^2, neither of which is in $\mathcal{D}_L$.

(a) Compute the adjoint L^*, along with the adjoint boundary conditions. Is L self adjoint?
(b) Compute the Green's function for L.
(c) Is L^{-1} compact? Justify your answer.

Problem 5. State and sketch a proof of the Weierstrass Approximation Theorem.