APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER

January 11, 2021

Applied Analysis Part, 2 hours

Name: ____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Recall that the DFT and inverse DFT are given by $\hat{y}_k = \sum_{j=0}^{n-1} y_j \bar{w}^{jk}$ and $y_j = \frac{1}{n}\sum_{j=0}^{n-1} \hat{y}_k w^{jk}$, where $w = e^{2\pi i/n}$.

(a) State and prove the Convolution Theorem for the DFT.

(b) Let a, x, y be column vectors with entries $a_0, \ldots, a_{n-1}$, $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$. In addition, let α, ξ and η be n-periodic sequences, the entries for one period, $k = 0, \ldots, n-1$, being those of a, x, and y, respectively. Consider the circulant matrix

$$A = \begin{pmatrix} a_0 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ a_1 & a_0 & a_{n-1} & \cdots & a_2 \\ a_2 & a_1 & a_0 & \cdots & a_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \end{pmatrix}.$$

Show that the matrix equation $Ax = y$ is equivalent to convolution $\eta = \alpha * \xi$.

(c) What are the eigenvalues and eigenvectors of A? Use parts (a) and (b) to prove your answer.

Problem 2. Let $Lu = -\frac{d^2u}{dx^2}$, $0 \le x \le 1$, with the domain of L given by

$$D_L := \{u \in L^2[0,1] : u'' \in L^2[0,1],\ u(0) = -u(1),\ u'(0) = -u'(1)\}.$$

(a) Show that L is self adjoint on $D(L)$.

(b) Find the Green's function $G(x,y)$ for the problem $Lu = f$, $u \in D_L$.

(c) Show that $Ku := \int_0^1 G(\cdot, y)u(y)dy$ is a compact self-adjoint operator.

(d) Without actually finding them, show that the eigenfunctions of L contain an orthonormal set that is complete in $L^2[0,1]$.

Problem 3. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Closed Range Theorem.

(b) Let $\mathcal{H} = L^2[0,1]$. Define the kernel $k(x,y) := x^3y^2$ and let $Ku(x) = \int_0^1 k(x,y)u(y)dy$. Show the K is in $\mathcal{C}(L^2[0,1])$.

(c) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (b) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0,1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Consider the functions ϕ and ψ defined below:

$$\phi(x) = \begin{cases} (|x|-1)^2(2|x|+1) & |x| \le 1 \\ 0 & |x| > 1 \end{cases}, \qquad \psi(x) = \begin{cases} x(|x|-1)^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}.$$

(a) Let $n \ge 2$ and $0 \le j \le n$. Show that the functions $\phi_j(x) := \phi(nx - j)$ and $\psi_j(x) := \frac{1}{n}\psi(nx-j)$ satisfy the following: $\phi_j(k/n) = \delta_{j,k}$, $\phi_j'(k/n) = 0$, $\psi_j(k/n) = 0$ and $\psi_j'(k/n) = \delta_{j,k}$.

(b) Use part (a) to show that the set $\{\phi_j, \psi_j\}_{j=0}^n$ forms a basis for the finite element space $S^{\frac{1}{3}}(3,1)$. You may assume that $\dim S^{\frac{1}{3}}(3,1) = 2n+2$.

(c) Use part (b) to define a (non-orthogonal) projection on $C^{(1)}[0,1]$.