Applied Analysis Part

January 10, 2023

Name: ____

Instructions: *Do any three problems. Show all work clearly. State the problem that you are skipping.* **No** *extra credit for doing all four.*

Problem 1. Let A be an $n \times n$ self-adjoint matrix, with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.

(a) State and prove the Courant-Fischer min-max theorem.

(b) Let $B = [b_1\ b_2\ b_3]$ be a real $n \times 3$ matrix, with b_1, b_2, b_3 being linearly independent. Assume that $\|x\| = 1$. If $q(x) = x^T Ax$ and $\widehat{q}(x) = q(x)|_{B^T x=0}$, show that

$$\lambda_1 \geq \max_{\|x\|=1} \widehat{q}(x) \geq \lambda_4.$$

Problem 2. Let $Lu = -(x^2 u')'$, $1 \leq x \leq 2$, with the domain of L given by

$$D_L := \{u \in L^2[1,2] : Lu \in L^2[1,2],\ u(1) = 0,\ u'(2) = 0\}.$$

The homogeneous solutions to $Lu = 0$ are x^{-1} and 1.

(a) Find the Green's function $g(x, y)$ for the problem $Lu = f$, $u \in D_L$.

(b) Show that $Ku := \int_0^1 g(\cdot, y)u(y)dy$ is a compact, self adjoint operator, and that 0 in not an eigenvalue of K.

(c) Without actually finding them, show that the eigenfunctions of L contain an orthonormal set that is complete in $L^2[1, 2]$.

Problem 3. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) State and prove the Fredholm Alternative.

(b) State the Closed Range Theorem.

(c) Let $\mathcal{H} = L^2[0, 1]$. Define the kernel $k(x, y) := x^3 y^2$ and let $Ku(x) = \int_0^1 k(x, y)u(y)dy$. Show that K is in $\mathcal{C}(\mathcal{H})$.

(d) Let $L = I - \lambda K$, $\lambda \in \mathbb{C}$, with K as defined in part (c) above. Find all λ for which $Lu = f$ *can* be solved for all $f \in L^2[0, 1]$. For these values of λ, find the resolvent $(I - \lambda K)^{-1}$.

Problem 4. Sketch a proof of the following: *If f is a piecewise C^1, 2π-periodic function, and if $S_N = \sum_{n=-N}^{N} c_n e^{inx}$ is the N^{th} partial sum of the Fourier series for f, then, for every $x \in \mathbb{R}$,*

$$\lim_{N\to\infty} S_N(x) = \frac{f(x^+) + f(x^-)}{2}.$$