Combined Applied Analysis/Numerical Analysis Qualifier
Numerical Analysis Part
August 5, 2022

Problem 1. Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain, $\boldsymbol{\beta} \in C^0(\overline{\Omega})^2$, $\mu \in C^0(\overline{\Omega})$ and $f \in L^2(\Omega)$. Assume that for some positive constants μ_0, μ_1, β_1, there holds $0 < \mu_0 \leq \mu(x) \leq \mu_1$ and $|\boldsymbol{\beta}(x)| \leq \beta_1$ for all $x \in \Omega$. In addition, we suppose that $\mathrm{div}(\boldsymbol{\beta}) = 0$. Consider the following weak formulation of a convection-diffusion problem: Seek $u \in H_0^1(\Omega)$ satisfying

$$a(u,v) := \int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega (\boldsymbol{\beta} \cdot \nabla u) v = \int_\Omega f v =: F(v), \qquad \forall v \in H_0^1(\Omega).$$

Accept that there exists a unique solution to the above problem.

(1) Given a shape-regular, quasi-uniform sequence of triangulations $\{\mathcal{T}_h\}_{h>0}$ of Ω (h denotes the largest outer circle diameter), we set

$$\mathbb{V}_h := \left\{ v_h \in H_0^1(\Omega) \mid v_h|_T \in \mathbb{P}^1(T), \quad \forall T \in \mathcal{T}_h \right\}$$

and define for $\alpha > 0$ the approximate bilinear form on $\mathbb{V}_h \times \mathbb{V}_h$

$$a_h(v_h, w_h) := a(v_h, w_h) + \alpha h \int_\Omega \nabla v_h \cdot \nabla w_h, \qquad \forall w_h, v_h \in \mathbb{V}_h.$$

Show that $a_h(v_h, v_h) \geq \mu_h \|\nabla v_h\|^2_{L^2(\Omega)}$ with $\mu_h := \mu_0 + \alpha h$ and deduce that the finite element formulation: Seek $u_h \in \mathbb{V}_h$ such that

$$a_h(u_h, v_h) = F(v_h) \qquad \forall v_h \in \mathbb{V}_h$$

has one and only one solution $u_h \in \mathbb{V}_h$.

(2) Show that for all $v_h \in \mathbb{V}_h$

$$\mu_h \|\nabla(v_h - u_h)\|^2_{L^2(\Omega)} \leq a_h(v_h, v_h - u_h) - F(v_h - u_h)$$

and prove that as a consequence

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \inf_{v_h \in \mathbb{V}_h} \left\{ \frac{1}{\mu_h} \sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} + \left(1 + \frac{M}{\mu_h}\right) \|\nabla(v_h - u)\|_{L^2(\Omega)} \right\},$$

where $M > 0$ denotes the continuity constant of $a(.,.)$, i.e.

$$a(w,v) \leq M \|\nabla w\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \qquad \forall v, w \in H_0^1(\Omega).$$

(3) Derive the estimate

$$\sup_{w_h \in \mathbb{V}_h} \frac{|a_h(v_h, w_h) - a(v_h, w_h)|}{\|\nabla w_h\|_{L^2(\Omega)}} \leq \alpha h \|\nabla v_h\|_{L^2(\Omega)}.$$

(4) Deduce the existence of a constant C such that

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq C \left(1 + \frac{M + \alpha}{\mu_h}\right) \|u\|_{H^2(\Omega)} h.$$

You can use, without proof, valid results for interpolation operators.

Problem 2. Let $\Omega \subset \mathbb{R}^2$ and $\{\mathcal{T}_h\}_{h>0}$ be a sequence of shape-regular and quasi-uniform triangulations of Ω. For $v \in H^1(\Omega)$, we define $\pi_h v \in L^2(\Omega)$ on each triangle $T \in \mathcal{T}_h$ by

$$\pi_h u|_T := \frac{1}{|T|} \int_T u \in \mathbb{R}.$$

Show that there exist a constant C independent on h such that for $v \in H^1(\Omega)$ there holds

$$\|v - \pi_h v\|_{L^2(\Omega)} \leq Ch|v|_{H^1(\Omega)}.$$

Hint: If needed, you can use without proof the Denis-Lions and Bramble-Hilbert Lemmas as well as the estimates relating norms on $T \in \mathcal{T}_h$ with norms on the reference triangle. Make sure to precisely state and check the assumptions of the results used.

Problem 3. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. Let $\{\mathcal{T}_h\}$ be a sequence of quasi-uniform (typical diameter $\sim h$) and shape-regular triangulations of Ω. We set

$$\mathbb{V}_h := \left\{ v_h \in H_0^1(\Omega) \mid v_h|_T \in \mathbb{P}^1(T), \quad \forall T \in \mathcal{T}_h \right\}$$

and let $\delta t > 0$. We consider the following *explicit Euler* time discretization of the heat equation: For $n \geq 1$, find $u_h^n \in \mathbb{V}_h$ recursively as satisfying

$$\int_\Omega \frac{u_h^{n+1} - u_h^n}{\delta t} v_h + \int_\Omega \nabla u_h^n \cdot \nabla v_h = 0, \qquad \forall v_h \in \mathbb{V}_h.$$

Show that for $n \geq 1$ there holds

$$\frac{1}{2\delta t} \int_\Omega |u_h^{n+1}|^2 + \left(1 - \frac{C^2 \delta t}{2h^2}\right) \sum_{j=0}^{n} \int_\Omega |\nabla u_h^j|^2 \leq \frac{1}{2\delta t} \int_\Omega |u_h^0|^2,$$

where C is the constant in the inverse estimate (which you can use without proof)

$$\|\nabla v_h\|_{L^2(\Omega)} \leq \frac{C}{h} \|v_h\|_{L^2(\Omega)}, \qquad \forall v_h \in \mathbb{V}_h.$$

Deduce a condition on the discretization parameters for the scheme to be stable.
Hint: You need to find two appropriate choices of test functions v_h to derive the stability estimate. Also recall that

$$2(a-b)a = a^2 - b^2 + (a-b)^2 \qquad \text{and} \qquad 2(a-b)b = a^2 - b^2 - (a-b)^2.$$