Applied/Numerical Analysis Qualifying Exam

August 9, 2012

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
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Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let ψ_j and ϕ_j, $j = 1, \ldots, n$, be in $L^2[0,1]$. Assume the sets $\{\psi_j\}_{j=1}^n$ and $\{\phi_j\}_{j=1}^n$ are linearly independent. Consider the kernel $\kappa(x,y) = \sum_{j=1}^n \psi_j(x)\overline{\phi}_j(y)$.

(a) Define the term *compact operator.*
(b) Show that the operator $Ku = \int_0^1 \kappa(\cdot, y)u(y)dy$ is compact on $L^2[0,1]$.
(c) State and sketch a proof for the Fredholm alternative for compact operators on a Hilbert space.
(d) With K as in part (b), show that the equation $(I - \lambda K)u = f$ has an L^2-solution for all $f \in L^2[0,1]$ if and only if $1/\bar{\lambda}$ is *not* an eigenvalue of the matrix A, where $A_{jk} = \langle \phi_j, \psi_k \rangle$.

Problem 2. Find the first term of the asymptotic series for $F(x) := \int_x^\infty t^x e^{-t} dt,\ x \to +\infty$.

Problem 3. Let $n > 2$ be an integer and let $x_j = j/n$, $j = 0, \ldots, n$. Consider the functional $J[y] = \frac{1}{2}\int_0^1 (y'')^2 dx$. The admissible functions are in $C^1[0,1]$. On each closed interval $[x_j, x_{j+1}]$, they are in $C^4[x_j, x_{j+1}]$, for $j = 0, \ldots, n-1$. Finally, for each j, $y(x_j) = y_j$ is fixed.

(a) Assume that the functional is Fréchet differentiable. Show that for $\eta \in C^2[0,1]$, $\eta(x_j) = 0$, $j = 0, \ldots, n$, one has

$$\Delta J[y,\eta] = \int_0^1 y^{(iv)}\eta dx + \sum_{j=0}^{n-1} y''\eta'\Big|_{x_j^+}^{x_{j+1}^-}.$$

(b) If the minimizer y of J exists, use the result above to show that y is a piecewise cubic spline that is in $C^2[0,1]$.

Problem 4. Let $f \in L^2(\mathbb{R})$. Use the following formulas for the Fourier transform and its inverse:

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \text{ and } f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}d\omega.$$

(a) Define the term *band-limited* function.
(b) Show that if f is band-limited, then it is infinitely differentiable on $\mathbb{R}$. (Actually, it's analytic.)
(c) State and prove the the Shannon Sampling Theorem.

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Cover Sheet – Numerical Analysis Part

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Problem 1. Consider the variational problem: find $u \in H^1(\Omega)$, such that $a(u,v) = L(v)$ for all $v \in H^1(\Omega)$, where $\Omega = (0,1) \times (0,1)$, Γ is its boundary, and

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx\, dy + \int_0^1 u(s,0)v(s,0)\, ds \quad \text{and} \quad L(v) = \int_\Gamma gv\, ds. \tag{1.1}$$

Let $V_h \subset H^1(\Omega)$ be a finite dimensional space of conforming piece-wise linear finite elements (Courant triangles) over regular partition of Ω into triangles. For continuous v, w defined on $\widetilde{\Gamma} \subseteq \Gamma$, let the bilinear form $Q_{\widetilde{\Gamma}}(v,w)$ come from the quadrature

$$Q_{\widetilde{\Gamma}}(v,w) = \sum_{e \subseteq \widetilde{\Gamma}} \frac{|e|}{2}\left(v(P_1^e)w(P_1^e) + v(P_2^e)w(P_2^e)\right) \approx \int_{\widetilde{\Gamma}} vw\, ds. \tag{1.2}$$

Here e is an edge of the triangulation of length $|e|$ with end points P_1^e and P_2^e. Consider the FEM: find $u_h \in V_h$ such that

$$a_h(u_h, v) = L_h(v), \ \forall v \in V_h, \tag{1.3}$$

where $a_h(u_h, v)$ and $L_h(v)$ are defined from $a(u_h, v)$ and $L(v)$ with the boundary integrals approximated using quadrature (1.2).

Complete the following tasks:

(a) **Derive** the strong form to the problem (1.1).
(b) **Prove** that the bilinear form $a(u,v)$ is **coercive** on H^1.
(c) **Prove** that for $\widetilde{\Gamma} = \{(x,0), 0 < x < 1\}$, there are constants c_1 and c_2, independent of h, such that

$$c_1 Q_{\widetilde{\Gamma}}(v,v) \le \int_0^1 v(x,0)^2\, dx \le c_2 Q_{\widetilde{\Gamma}}(v,v), \ \forall v \in V_h.$$

Note that this inequality and part (b) immediately imply

$$a_h(v,v) \ge \alpha \|v\|^2_{H^1(\Omega)}, \ \forall v \in V_h$$

for some $\alpha > 0$ independent of h.

(d) Apply Strang's First Lemma to **estimate the error** in H^1-norm for the FEM (1.3). You may assume that g is as regular (smooth) as needed by your analysis and you can use (without proof) standard approximation properties for the finite element space V_h.

Problem 2. Consider the following initial boundary value problem: find $u(x,t)$ such that

$$\begin{aligned} \frac{\partial}{\partial t}(u - \Delta u) - \mu \Delta u &= f, \ x \in \Omega, \ T \ge t > 0, \\ u(x,t) &= 0, \ x \in \partial\Omega, \ T \ge t > 0, \\ u(x,0) &= u_0(x), \ x \in \Omega, \end{aligned} \tag{2.1}$$

where Ω is a polygonal domain in $\mathcal{R}^2$, $\mu > 0$ is a given constant, and $f(x,t)$ and $u_0(x)$ are given right hand side and initial data functions.

(a) **Derive** a weak formulation of this problem and derive an *a priori* estimate for the solution in the norm

$$\|u(t)\|_{H^1(\Omega)} = \left(\|u(t)\|^2_{L^2(\Omega)} + \|\nabla u(t)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \tag{2.2}$$

in terms of the right-hand side and the initial data.

(b) **Write down** the fully discrete scheme based on implicit (backward) Euler approximation in time and the finite element method in space with continuous piece-wise linear functions. **Prove** unconditional stability in the H^1-norm for the resulting approximation.

(c) Consider now the forward Euler approximation for the derivative in t. **Find** the Courant condition for stability of the resulting method in a norm of your choice.

Problem 3. Let $\mathcal{T}_h$ be a partition of $(0,1)$ into finite elements of equal size $h = 1/N$, $N > 1$ an integer, and $x_i = ih$, $i = 0, 1, \ldots, N$. Consider the finite dimensional space V_h of continuous **piece-wise quadratic** functions on $\mathcal{T}_h$. The degrees of freedom on finite element (x_{i-1}, x_i) are

$$\left\{ v(x_{i-1}),\ v(x_i),\ \frac{1}{h}\int_{x_{i-1}}^{x_i} v\,dx \right\}. \tag{3.1}$$

(1) **Explicitly find the nodal basis** of V_h over the finite element (x_{i-1}, x_i), corresponding to these degrees of freedom.

(2) **Prove** that

$$\sup_{\phi \in H^1(0,1)} \frac{\displaystyle\int_0^1 (u - \Pi_h u)\phi\,dx}{\|\phi\|_{H^1(0,1)}} \le Ch\,\|u - \Pi_h u\|_{L^2(0,1)}, \quad \forall u \in H^1(0,1).$$

Here $\Pi_h u$ is the finite element interpolant of u with respect to the nodal basis of V_h defined by (3.1).