Applied Mathematics Qualifying Exam

May 22, 2006

Instructions: *Do any 7 of the 9 problems in this exam. Show all of your work clearly. Please indicate which 2 of the 9 problems you are skipping.*

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem in such a way that it becomes trivial.

1. Let $\{\phi_n(x)\}_{n=0}^{\infty}$ be a set of polynomials orthogonal with respect to a weight function $w(x)$ on a domain $[a, b]$. Assume that the degree of ϕ_n is n, and that coefficient of x^n in $\phi_n(x)$ is $k_n > 0$.

   (a) Show that ϕ_n is orthogonal to all polynomials of degree $n - 1$ or less.

   (b) Show that the set $\{\phi_n(x)\}_{n=0}^{\infty}$ is the same, up to multiples, as the one gotten by using the Gram-Schmidt process.

   (c) Show that the polynomials satisfy the recurrence relation below; find A_n in terms of the k_n's.

   $$\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n\phi_{n-1}(x)$$

2. Let $\mathcal{H}$ be a complex Hilbert space, with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ being the inner product and norm.

   (a) State and prove the Projection (Decomposition) Theorem

   (b) State and prove the Fredholm Alternative. (Hint: use the Projection Theorem.)

   (c) Take $\mathcal{H} = L^2[0, 1]$. Let $k(x, y) = xy^3$ and let $Lu(x) = u(x) + \lambda \int_0^1 k(x, y)u(y)dy$, where $\lambda \in \mathbb{R}$. Find the adjoint of L. Determine when $Lu = f$ has a solution. (You may assume that L has closed range.)

3. Among all curves $y = f(x)$, $-1 \le x \le 1$, $f(-1) = f(1) = 0$, with fixed length $L > 2$, find the one that minimizes the surface area formed when the curve is rotated about the x-axis.

4. Consider $I_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos(\theta)}d\theta$.

   (a) Show that I_0 satisfies $xy'' + y' - xy = 0$, which is Bessel's modified equation of order 0 .

   (b) Obtain the asymptotic formula $I_0(x) \sim \dfrac{e^x}{\sqrt{2\pi x}}$, as $x \to \infty$.

5. Let $\mathcal{H}$ be a complex Hilbert space, with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ being the inner product and norm.

   (a) Define these terms (all for $\mathcal{H}$): bounded linear operator, adjoint of a bounded linear operator, compact operator.

   (b) Show that if L is a bounded, self-adjoint linear operator on $\mathcal{H}$, then $\|L\| = \sup_{\|u\|=1} |\langle Lu, u\rangle|$. (Hint: look at $\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle$.)

   (c) Prove this: *If K is a compact, self-adjoint linear operator on a Hilbert space $\mathcal{H}$, then either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

6. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions. You are given that polynomials are in $\mathcal{S}'$ and that $\psi = \phi'$ for some $\phi \in \mathcal{S}$ if and only if $\int_{-\infty}^{\infty} \psi(x)dx = 0$.

   (a) Define $\mathcal{S}$ and give the semi-norm topology for it. In addition, define $\mathcal{S}'$.

   (b) Show that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k\widehat{T}$, where $k = 1, 2, \ldots$. (Use the Fourier transform convention $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{i\omega t}dt$.)

   (c) Prove: *Let $T \in \mathcal{S}'$ and $n \in \mathbb{Z}_+$. Then $\omega^{n+1}\widehat{T} = 0$ if and only if T is a polynomial of degree n.* (Hint: use induction.)

7. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u(0) = 0\}.$$

(a) Find the Green's function $G(x,\xi;z)$ for $-G''-zG=\delta(x-\xi)$, with $G(0,\xi;z)=0$. (This is the kernel for the resolvent $(L-zI)^{-1}$.)

(b) Employ the spectral theorem to obtain the sine transform formulas,

$$F(\mu)=\int_0^\infty f(x)\sin(\mu x)dx \text{ and } f(x)=\frac{2}{\pi}\int_0^\infty F(\mu)\sin(\mu x)d\mu\,.$$

8. Do *one* of the following.

(a) State and prove the Contraction Mapping Theorem.

(b) Sketch a proof of the following: *If f is a piecewise C^1, 2π-periodic function, and if $S_N=\sum_{n=-N}^{N} c_n e^{inx}$ is the N^{th} partial sum of the Fourier series for f, then, for every $x\in\mathbb{R}$,*

$$\lim_{N\to\infty} S_N(x)=\frac{f(x^+)+f(x^-)}{2}.$$

(c) Sketch a proof of this 1D Sobolev theorem: *If u is a distribution having a distributional derivative u' in $L^2[a,b]$, then $u\in C[a,b]$ and there is a constant $C>0$ that depends only on $b-a$ for which*

$$\|u\|_{C[a,b]}\le C\|u\|_{H^1[a,b]},\quad \|u\|_{H^1[a,b]}^2:=\int_a^b(|u|^2+|u'|^2)dx.$$

9. Let $p\in C^{(2)}[0,1]$, and $q,w\in C[0,1]$, with $p,q,w>0$. Consider the Sturm-Liouville (SL) eigenvalue problem, $(p\phi')'-q\phi+\lambda w\phi=0$, subject to $\phi(0)=0$ and either (A) $\phi(1)=0$ or (B) $\phi'(1)+\phi(1)=0$. In addition, for $\phi\in C^{(1)}[0,1]$, let $D[\phi]:=\int_0^1(p\phi'^2+q\phi^2)dx$ and $H[\phi]:=\int_0^1 w\phi^2dx$.

(a) Show that minimizing the functional $D[\phi]$, subject to the constraint $H[\phi]=1$ and boundary conditions $\phi(0)=\phi(1)=0$, yields the SL problem (A).

(b) State the variational problem that will yield the SL problem (B). Verify that your answer is correct by calculating the variational (Fréchet) derivative and setting it equal to 0.

(c) State the MINIMAX Principle. Use it to show that the n^{th} eigenvalue of the SL problem (A) is larger than or equal to the n^{th} eigenvalue of the SL problem (B).