# Applied/Numerical Analysis Qualifying Exam

January 11, 2010

**Policy on misprints:** The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

## Part 1: Applied Analysis

**Instructions:** *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

1. Let $Lu = \frac{d}{dx}\left((1+x)\frac{du}{dx}\right)$. Find the Green's function for $Lu = f$, $u(0) = 0$ and $u'(1) = 0$.

2. This problem concerns Mallat's multiresolution analysis (MRA).

   (a) Define the term *multiresolution analysis.* For the Haar MRA, state the scaling function $\phi$, the wavelet $\psi$, the approximation spaces $V_j$, the dilation (or scaling) relation, and the wavelet spaces $W_j$.

   (b) Use the scaling and wavelet coefficients given below to derive the *decomposition* and *reconstruction* formulas for the Haar MRA.

   $$s_k^j = 2^j \int_{\mathbb{R}} f(x)\phi(2^j x - k)dx \text{ and } d_k^j = 2^j \int_{\mathbb{R}} f(x)\psi(2^j x - k)dx.$$

   (c) Let $f$ be compactly supported and continuous on $\mathbb{R}$. Show that $s_k^j$ is the average of $f(x)$ over the interval $[k \cdot 2^{-j}, (k+1) \cdot 2^{-j}]$, where $s_k^j$ is given in part 2b. What role does this formula play in the initialization step of a wavelet analysis? (One or two sentences will suffice.)

3. A chain having uniform linear density $\rho = 1$ hangs between the points (-1,0) and (1,0). (The positive $y$ direction is downward; the acceleration due to gravity is $g = 1$.) The total mass $m$, which is fixed, and the total energy $E$ of the chain are

$$m = \int_{-1}^{1} \sqrt{1 + y'^2}dx > 2 \text{ and } E[y] = \int_{-1}^{1} y\sqrt{1 + y'^2}dx$$

Assuming that the chain hangs in a shape that minimizes the energy, find the shape of the hanging chain. (Hint: the integrand of the functional to be minimized doesn't depend on $x$.)

4. Let $\mathcal{H}$ be a complex (separable) Hilbert space, with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ being the inner product and norm.

   (a) Let $\lambda \in \mathbb{C}$ be fixed. If $K : \mathcal{H} \to \mathcal{H}$ is a compact linear operator, show that the range of the operator $L = I - \lambda K$ is closed.

   (b) Briefly explain why the operator $Ku(x) := \int_0^1 (3 + 4xy^2)u(y)dy$ is compact on $\mathcal{H} = L^2[0, 1]$. Determine the values of $\lambda \in \mathbb{C}$ for which $u = f + \lambda K u$ has a solution for all $f \in L^2[0, 1]$. State the theorem that you are using to answer the question.

## Part 2: Numerical Analysis

**Instructions:** *Do all problems in this part of the exam. Show all of your work clearly.*

1. Consider the system

$$\begin{aligned} -\Delta u - \phantom{\Delta}\phi &= f \\ u \phantom{\Delta} - \Delta\phi &= g \end{aligned} \tag{1}$$

in the bounded, smooth domain $\Omega$, with boundary conditions $u = \phi = 0$ on $\partial\Omega$.

   (a) Derive a weak formulation of the system (1), using suitable test functions for each equation. Define a bilinear form $a\big((u,\phi),(v,\psi)\big)$ such that this weak formulation amounts to

$$a\big((u,\phi),(v,\psi)\big) = (f,v) + (g,\psi). \tag{2}$$

   (b) Choose appropriate function spaces for $u$ and $\phi$ in (2).

   (c) Show, that the weak formulation (2) has a unique solution. Hint: Lax-Milgram.

   (d) For a domain $\Omega_d = (-d,d)^2$, show that

$$\|u\|^2 \le cd^2\|\nabla u\|^2 \tag{3}$$

   holds for any function $u \in H_0^1(\Omega_d)$.

   (e) Now change the second "-" in the first equation of (1) to a "+". Use (3) to show stability for the modified equation on $\Omega_d$, provided that $d$ is sufficiently small.

2. Consider the two finite elements $(\tau, Q_1, \Sigma)$ and $(\tau, \widetilde{Q}_1, \Sigma)$, where $\tau = [-1,1]^2$ is the reference square and

$$\begin{aligned} Q_1 &= \operatorname{span}\big\{1, x, y, xy\big\}, \\ \widetilde{Q}_1 &= \operatorname{span}\big\{1, x, y, x^2 - y^2\big\}. \end{aligned}$$

$\Sigma = \{w(-1,0), w(1,0), w(0,-1), w(0,1)\}$ is the set of the values of a function $w(x,y)$ at the midpoints of the edges of $\tau$.

(a) Which of the two elements is unisolvent? Prove it!

(b) Show that the unisolvent element leads to a finite element space, which is not $H^1$-conforming.

3. Consider the following initial boundary value problem: find $u(x, t)$ such that

$$\begin{aligned} u_t - u_{xx} + u &= 0, \qquad 0 < x < 1, \quad t > 0 \\ u_x(0, t) = u_x(1, t) &= 0, \qquad t > 0 \\ u(x, 0) &= g(x), \qquad 0 < x < 1. \end{aligned}$$

(a) Derive the semi-discrete approximation of this problem using linear finite elements over a uniform partition of $(0, 1)$. Write it as a system of linear ordinary differential equations for the coefficient vector.

(b) Further, derive discretizations in time using backward Euler and Crank-Nicolson methods, respectively.

(c) Show that both fully discrete schemes are unconditionally stable with respect to the initial data in the spatial $L^2(0, 1)$-norm.