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On the Norms and Minimal Properties of de la Vallée Poussin's type Operators

Beata Deregowska · **Simon Foucart** · **Barbara Lewandowska** · **Lesław Skrzypek**

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Abstract In this paper we consider the de la Vallée Poussin's type operators $H_{rn,sn}$

$$H_{rn,sn} := \frac{F_{rn} + F_{rn+1} + ... + F_{sn-1}}{(s-r)n},$$

where F_k are classical Fourier projections onto Π_k (the space of trigonometric polynomials of degree less than or equal to k). We determine when $H_{n,sn}$ is the minimal generalized projection and provide the asymptotic behavior of the norm $\|H_{n,sn}\|$. Additionally, we contrast the results obtained for the trigonometric system to the results obtained for the Rademacher system.

Keywords de la Vallée Poussin's operators · Minimal projections

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1 Introduction

The symbol $\mathscr{L}(X,Z)$ stands for the space of all linear continuous operators from a normed space X into a normed space Z. Let $V \subset Z$ be (closed) subspaces of a Banach

Beata Deregowska
Pedagogical University of Cracow, Institute of Mathematics, Podchorazych 2, 30-084 Krakow, Poland
E-mail: bderegowska@up.krakow.pl

Simon Foucart
Department of Mathematics, Texas A&M University, College Station, TX 77843-3368
E-mail: foucart@tamu.edu

Barbara Lewandowska
Faculty of Mathematics and Computer Science, Jagiellonian University, Lojasiewicza 6, 30-048 Krakow, Poland
E-mail: barbara.lewandowska@im.uj.edu.pl

Lesław Skrzypek
Department of Mathematics and Statistics, University of South Florida, 4202 E. Fowler Ave., CMC 342, Tampa, FL 33620-5700
E-mail: skrzypek@usf.edu

space X. Put

$$\mathscr{P}_V(X,Z) := \{P \in \mathscr{L}(X,Z) : P|_V = id\}. \tag{1}$$

Each element of $\mathscr{P}_V(X,Z)$ is called a **generalized projection**. We say that P_o is a **minimal generalized projection** (MGP) if it has the smallest possible norm, that is if

$$\|P_o\| = \lambda_Z(V,X) := \inf\{\|P\| : P \in \mathscr{P}_V(X,Z)\}. \tag{2}$$

The case $V = Z$ corresponds to standard projections and we denote $\mathscr{P}(X,V) := \mathscr{P}_V(X,V)$ and $\lambda(V,X) := \lambda_V(V,X)$ for simplicity. Projections play an important role in numerical analysis. Given a projection P, we can approximate x by Px and the error of such approximation $\|x - Px\|$ can be estimated by means of the elementary inequality

$$\|x - Px\| \leq \|id - P\| \cdot \text{dist}(x,V) \leq (1 + \|P\|) \cdot \text{dist}(x,V), \tag{3}$$

where $\text{dist}(x,V) := \inf\{\|x - v\| : v \in V\}$. From the above inequality we deduce that the smaller the norm of P the better the approximation of x by Px. Notice that inequality (3) also holds for generalized projections. Moreover, if $V \subset Z$ then $\mathscr{P}_V(X,V) \subset \mathscr{P}_V(X,Z)$. The question arises, to what extent we can improve the approximation of x by Px if we consider generalized projections. The related question is how smaller the norms of the generalized projections are in relation to $\lambda(V,X)$.

The classic example considers the Fourier projection. Let $X = \mathscr{C}_{2\pi}(\mathbb{R})$ (the space of continuous 2π-periodic functions on $\mathbb{R}$) or $X = L_1[0,2\pi]$ and denote by Π_n the space of all trigonometric polynomials of a degree less than or equal to n. The Fourier projection $F_n : X \to \Pi_n$ (that is the orthogonal projection onto Π_n) is given by

$$F_n(f)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(s) D_n(t-s)\, ds, \tag{4}$$

where D_n is a Dirichlet kernel

$$D_n(t) = 1 + 2\cos(t) + \ldots + 2\cos(nt) = \frac{\sin(\frac{2n+1}{2}t)}{\sin(\frac{t}{2})}. \tag{5}$$

Observe that, the Fourier projection is a self-adjoint operator so its operator norm induced by L_1 norm and L_∞ (or supremum) norm is the same. The Fourier projection is uniquely minimal ([13], [2] and [10]) and its norm is given by ([7] and [9])

$$\|F_n\| = \|D_n\| \approx \frac{4}{\pi^2} \ln n. \tag{6}$$

The main difficulty here is that $\|F_n\| \to \infty$ which, by Banach-Steinhaus theorem, means that the set of $f \in X$ for which the sequence $F_n(f)$ diverges is dense in X. We also refer the reader to [22], [8], [11], [12] and [18] for further reading regarding the properties of Fourier projection.

In 1918 de la Vallée Poussin ([15]) introduced the generalized projections considering the arithmetic means of Fourier projections

$$H_{n,2n} := \frac{F_n + F_{n+1} + \ldots + F_{2n-1}}{n} \in \mathscr{P}_{\Pi_n}(X, \Pi_{2n-1}). \tag{7}$$

Observe that, the de la Vallée Poussin projection is a self-adjoint operator so its operator norm induced by L_1 norm and L_∞ (or supremum) norm is the same. The main advantage of de la Vallée Poussin generalized projections $H_{n,2n}$ is that their norms are uniformly bounded. Recently, it has been shown that $H_{n,2n}$ are in fact the generalized minimal projections ([5]) and their exact norms have been established ([14])

$$\|H_{n,2n}\| = \|F_1\| = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} = 1.435991.... \tag{8}$$

We also refer the reader to [3], [4], [20] and [21] for further reading regarding the properties of de la Vallée Poussin operators.

In this paper we consider the de la Vallée Poussin's type operators $H_{n,sn}$

$$H_{n,sn} := \frac{F_n + F_{n+1} + ... + F_{sn-1}}{n} \in \mathscr{P}_{\Pi_n}(X, \Pi_{sn-1}). \tag{9}$$

We provide a full characterization when $H_{n,sn}$ is the minimal generalized projection (see Theorems 12, 14 and 16)

Theorem 1 *$H_{n,sn}$ is (MGP) in $\mathscr{P}_{\Pi_n}(X, \Pi_{sn-1})$ if and only if $s = 1$ or $s = 2$.*

We can easily see that $\|H_{n,sn}\| \to 1$ as $s \to \infty$. But how fast is the convergence? Or how much do we gain by setting the overspace Z to be Π_{sn-1} (instead of Π_{2n-1})? We provide the tight asymptotic behavior of the L_1 norm and L_∞ (or supremum) norm of the de la Vallée Poussin operators (see Theorem 18 given on page 13).

Theorem 2 $\|H_{n,sn}\| = 1 + \Theta(\frac{1}{s})$*, for all* $n \in \mathbb{N}$.

Here $a_n = \Theta(b_n)$ denotes that a_n is bounded both above and below by b_n asymptotically.

Summarizing, for the trigonometric system, we considered the sequence of subspaces Π_n and the sequence of corresponding overspaces such that

$$\Pi_n \subset \Pi_{n,2n-1} \subset ... \subset \Pi_{n,sn-1} \subset ... \subset X \tag{10}$$

and

$$\lambda_{\Pi_n}(X, \Pi_n) \approx \frac{4}{\pi^2} \ln n \tag{11}$$

$$\lambda_{\Pi_n}(X, \Pi_{2n-1}) = \frac{1}{3} + \frac{2\sqrt{3}}{\pi} = 1.435991... \tag{12}$$

$$\lambda_{\Pi_n}(X, \Pi_{sn-1}) = 1 + \Theta(\frac{1}{s}) \tag{13}$$

We contrast the results obtained for the trigonometric system to the results obtained for the Rademacher system. Let $X = L_1[0,1]$ and denote by $Rad_n = span\{r_0, ..., r_{n-1}\}$ the subspace spanned by the first n Rademacher functions. The Rademacher projection $R_n : X \to Rad_n$ (that is the orthogonal projection into Rad_n) is given by

$$R_n(f)(t) = \int_0^1 f(s) D_A^r(t \oplus s)\, ds, \tag{14}$$

where $D_n^r = \sum_{i=0}^{n-1} r_i$ is a corresponding Dirichlet kernel. The Rademacher projection is minimal ([1]) and the behavior of its norm is given by ([17])

$$\|R_n\| = \Theta(\sqrt{n}). \tag{15}$$

Interestingly, and in contrast to the trigonometric system, we cannot improve anything by considering the generalized projections onto Rad_m (for $m > n$) as we obtain the following result (see Theorem 23).

Theorem 3 *R_n is (MGP) in $\mathscr{P}_{Rad_n}(L_1[0,1], Rad_m)$.*

Furthemore, if we set the overspace Z to be S_n (the space of simple functions generated by characteristic functions of the intervals $I_{l,n} = [\frac{l}{2^n}, \frac{l+1}{2^n})$, where $l \in \{0,1,...,2^n - 1\}$) and consider the projection $L_n \in \mathscr{P}_{Rad_n}(L_1[0,1], S_n)$ given by

$$L_n(f) = \sum_{l=0}^{2^n-1} \left(\int_{I_{l,n}} f(x)dx \right) \chi_{I_{l,n}} \tag{16}$$

we establish the result below

Remark 4 *$\|L_n\| = 1$ and, therefore, L_n is (MGP) in $\mathscr{P}_{Rad_n}(L_1[0,1], S_n)$.*

The following well known characterization of best approximation is a direct consequence of Hahn-Banach theorem and will be our main tool in determining whether a given generalized projection is minimal or not.

Theorem 5 ([19], Theorem 1.1) *Let X be a Banach space and $V \subset X$ be a linear subspace and let $x_0 \in X \setminus \mathrm{cl}(V)$. Then v_0 is a best approximation to x_0 in V iff there exists $f \in S(X^*)$ such that*

$$f(x_0 - v_0) = \|x_0 - v_0\| \quad \text{and} \quad f|_V \equiv 0. \tag{17}$$

2 De la Vallée Poussin's type operators

Definition 6 *Let $X = \mathscr{C}_{2\pi}(\mathbb{R})$ or $X = L_1[0,2\pi]$. Let $H_{rn,sn} : X \to \Pi_{sn-1}$ be given by*

$$H_{rn,sn}(f)(t) = \frac{1}{(s-r)n} \sum_{k=rn}^{sn-1} F_k(f)(t) = \frac{1}{2\pi} \int_0^{2\pi} f(u) V_{rn,sn}(t-u) du, \tag{18}$$

where

$$\begin{aligned} V_{rn,sn}(t) &= \frac{1}{(s-r)n} \sum_{k=rn}^{sn-1} D_k(t) = \frac{\sin(\frac{s-r}{2}nt)\sin(\frac{s+r}{2}nt)}{(s-r)n(\sin(\frac{1}{2}t))^2} \\ &= \frac{\sin^2(\frac{snt}{2}) - sin^2(\frac{rnt}{2})}{(s-r)n(\sin(\frac{1}{2}t))^2}. \end{aligned} \tag{19}$$

for some $r,s \in \mathbb{N}$ such that $r < s$. Then $H_{rn,sn} \in \mathscr{P}_{\Pi_{rn}}(X, \Pi_{sn-1})$ is a generalized projection called de la Vallée Poussin's type operator.

Observe that for every operator P which is de la Vallée Poussin's type operator there exist unique natural numbers r, s, n such that r and s are relatively prime and $P = H_{rn,sn}$.

Theorem 7 ([14], Theorem 1.1) *Let $r, s, n \in \mathbb{N}$. Then*

$$\|H_{rn,sn}\| = \|H_{r,s}\|. \tag{20}$$

PROOF. Using the identity

$$\frac{1}{\sin^2(t)} = \sum_{k=-\infty}^{+\infty} \frac{1}{(t+k\pi)^2} \tag{21}$$

we can provide a different and much shorter proof of the statement of this theorem. Indeed, by (21) we obtain

$$\begin{aligned} H_{rn,sn}(t) &= \frac{1}{2\pi}\int_0^{2\pi} f(u)V_{rn,sn}(t-u)du = \frac{1}{2\pi}\int_0^{2\pi} f(t-u)V_{rn,sn}(u)du \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(t-u)\frac{1}{(s-r)n}\sum_{k=-\infty}^{+\infty}\frac{\sin^2(\frac{snu}{2})-\sin^2(\frac{rnu}{2}))}{(u+2k\pi)^2}\,du. \end{aligned}$$

Since the above series is uniformly convergent on $\mathbb{R}$ there follows

$$\begin{aligned} H_{rn,sn}(t) &= \frac{1}{2\pi}\int_0^{2\pi} f(t-u)\frac{1}{(s-r)n}\sum_{k=-\infty}^{+\infty}\frac{\sin^2(\frac{snu}{2})-\sin^2(\frac{rnu}{2}))}{(u+2k\pi)^2}\,du \\ &= \frac{1}{2\pi(s-r)n}\sum_{k=-\infty}^{+\infty}\int_0^{2\pi} f(t-u)\frac{\sin^2(\frac{snu}{2})-\sin^2(\frac{rnu}{2}))}{(u+2k\pi)^2}\,du \\ &= \frac{1}{2\pi(s-r)n}\sum_{k=-\infty}^{+\infty}\int_0^{2\pi} f(t-u-2k\pi)\frac{\sin^2(\frac{sn(u+2k\pi)}{2})-\sin^2(\frac{rn(u+2k\pi)}{2}))}{(u+2k\pi)^2}\,du \\ &= \frac{1}{2\pi(s-r)n}\sum_{k=-\infty}^{+\infty}\int_{2k\pi}^{2(k+1)\pi} f(t-u)\frac{\sin^2(\frac{snu}{2})-\sin^2(\frac{rnu}{2}))}{u^2}\,du \\ &= \frac{1}{2\pi(s-r)n}\int_{-\infty}^{+\infty} f(t-u)\frac{\sin^2(\frac{snu}{2})-\sin^2(\frac{rnu}{2}))}{u^2}\,du \\ &= \frac{2}{\pi(s-r)n}\int_{-\infty}^{+\infty} f(t-u)\frac{\sin(\frac{s-r}{2}nu)\sin(\frac{s+r}{2}nu)}{u^2}\,du \end{aligned}$$

Hence

$$\|H_{rn,sn}\| = \frac{2}{\pi(s-r)n}\int_{-\infty}^{+\infty}\frac{|\sin(\frac{s-r}{2}nu)\sin(\frac{s+r}{2}nu)|}{u^2}du. \tag{22}$$

Making the substitution $\frac{2}{(s-r)n}x = u$, we get

$$\|H_{rn,sn}\| = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{|\sin x\sin(\frac{s+r}{s-r}x)|}{x^2}dx,$$

which shows that the norm of the operator $H_{rn,sn}$ is independent of n.

Corollary 8 *Let $p,q\in\mathbb{N}$ such that $p>q$ then*

$$\int_{-\infty}^{+\infty}\frac{|\sin x\sin(\frac{p}{q}x)|}{x^2}dx=\pi\|H_{p-q,p+q}\|.$$

Lemma 9 ([5], Lemma 2.1) *Let X be $\mathscr{C}_{2\pi}(\mathbb{R})$ or $L_1[0,2\pi]$. Let $\emptyset\neq A\subset B$ be finite subsets of $\mathbb{Z}$. We define $V:=\operatorname{span}\{e^{ikt}:\ k\in A\}$ and $Z:=\operatorname{span}\{e^{ikt}:\ k\in B\}$. For every $P\in\mathscr{P}_V(X,Z)$ there exists $\widetilde{P}\in\mathscr{P}_V(X,Z)$ such that*

$$\|\widetilde{P}\|\leq\|P\|\ \ and\ \ \widetilde{P}(f)(t)=\frac{1}{2\pi}\int_0^{2\pi}f(s)\sum_{k\in B}a_ke^{ik(t-s)}ds \tag{23}$$

for some $a_k\in\mathbb{C}$, such that $a_k=1$ for $k\in A$.

Lemma 10 *Let X be $\mathscr{C}_{2\pi}(\mathbb{R})$ or $L_1[0,2\pi]$. Then $H_{rn,sn}$ is (MGP) in $\mathscr{P}_{\Pi_{rn}}(X,\Pi_{sn-1})$ iff for all $k\in\mathbb{Z}$ such that $rn<|k|<sn$ the following holds*

$$\int_0^{2\pi}\operatorname{sgn}(V_{rn,sn}(t))e^{ikt}\,dt=0.$$

Proof. By *Lemma 9* it is enough to show that $H_{rn,sn}$ is minimal in the set of generalized projections given by the convolutions (23). If P is a generalized projection of this form then there exists $b_k\in\mathbb{C}$, such that

$$P(f)(t)=\frac{1}{2\pi}\int_0^{2\pi}f(s)\Big(V_n(t-u)+\sum_{rn<|k|<sn}b_ke^{ik(t-u)}\Big)du.$$

Then

$$\|P\|=\frac{1}{2\pi}\int_0^{2\pi}\Big|V_n(t)+\sum_{rn<|k|<sn}b_ke^{ikt}\Big|dt.$$

Hence $H_{rn,sn}$ is MGP if and only if 0 is a best approximation to the function $V_{rn,sn}$ in $Y:=\operatorname{span}\{e^{ikt}:|k|\in\{rn+1,\ldots,sn-1\}\}$ equipped with L_1-norm. Since $sgn(V_{rn,sn})\in L_\infty[0,2\pi]$ is a unique norming functional for $V_{rn,sn}\in L_1[0,2\pi]$, by *Theorem 5* we obtain the assertion. □

Lemma 11 *Let $n,s,r\in\mathbb{N}$ and $s>r$. Then the function $\operatorname{sgn}(V_{rn,sn}):(0,2\pi)\to\mathbb{R}$ (see (19)) has the following properties*

1. *$\operatorname{sgn}(V_{rn,sn})$ is a $\frac{2\pi}{n}$ periodic function;*
2. *$\operatorname{sgn}(V_{rn,sn}(\frac{\pi}{n}-t))=\operatorname{sgn}(V_{rn,sn}(\frac{\pi}{n}+t))$ for all $t\in(0,\frac{\pi}{n})$;*
3. *zeros of the function $\operatorname{sgn}(V_{rn,sn})$ in the interval $(0,\frac{\pi}{n})$ have one of two forms:*
   (i) $\frac{2\pi a}{(s+r)n}$ for $a\in\{1,\ldots,p\}$, where p is the biggest natural number which is less than $\frac{s+r}{2}$;
   (ii) $\frac{2\pi a}{(s-r)n}$ for $a\in\{1,\ldots,q\}$, where q is the biggest natural number which is less than $\frac{s-r}{2}$.

PROOF. Let $f(t) := \sin(\frac{s+r}{2}nt)\sin(\frac{s-r}{2}nt)$ for $t \in (0, 2\pi)$. By (19) we have $\mathrm{sgn}(V_{rn,sn}) = \mathrm{sgn}(f)$. Observe that, if the function f has the above properties, $\mathrm{sgn}(f)$ has the same properties.
1. Fix $t \in (0, \frac{2\pi(n-1)}{n})$, then $f(t+\frac{2\pi}{n}) = \sin(\frac{s+r}{2}nt+(s+r)\pi)\cdot\sin(\frac{s-r}{2}nt+(s-r)\pi) = (-1)^{s+r}(-1)^{s-r}f(t) = f(t)$.
2. Let $t \in (0, \frac{\pi}{n})$, then $f(\frac{\pi}{n}-t) = f(t-\frac{\pi}{n}) = (-1)^{s+r}(-1)^{s-r}f(t-\frac{\pi}{n}) = \sin(\frac{s+r}{2}n(t-\frac{\pi}{n})+(s+r)\pi)\cdot\sin(\frac{s-r}{2}n(t-\frac{\pi}{n})+(s-r)\pi) = f(\frac{\pi}{n}+t)$.
3. Follows from the definition of the function f. □

For simplicity of notation, we denote $H^1_{rn} := H_{rn,(r+1)n}$, $V^1_{rn} := V_{rn,(r+1)n}$ and $H^2_{rn} := H_{rn,(r+2)n}$, $V^2_{rn} := V_{rn,(r+2)n}$.

Theorem 12 *Let $X = \mathscr{C}_{2\pi}(\mathbb{R})$ or $X = L_1[0,2\pi]$. Then for all $r \in \mathbb{N}$ operator H^1_{rn} is MGP in $\mathscr{P}_{\Pi_{rn}}(X, \Pi_{(r+1)n-1})$.*

PROOF. By *Lemma 10* it is sufficient to show that, for every $k \in \mathbb{Z}$, such that $|k| \in (rn, (r+1)n)$ we have $\int_0^{2\pi}\mathrm{sgn}(V^1_{rn}(t))e^{ikt}dt = 0$. According to *Lemma 11,* $V^1_{rn}(t) = 0$ if and only if $t = \frac{2\pi b}{(2r+1)n}$ for $b \in \{1, \ldots, (2r+1)n-1\}$. Moreover, only at double zeros of V^1_{rn} ($t = \frac{2\pi a}{n}$ for $a \in \{1, \ldots n-1\}$) the function does not change the sign. Denote $I_b = \left(\frac{2\pi b}{(2r+1)n}, \frac{2\pi(b+1)}{(2r+1)n}\right)$ for $b \in \{0, \ldots, (2r+1)n-1\}$. Observe that between two double zeros there is always an even number of simple zeros, which implies that the function V^1_{rn} changes the sign an even number of times between them. Hence $\mathrm{sgn}(f)|_{I_b} = \mathrm{sgn}(f)|_{(0,\frac{2\pi}{(2r+1)n})} = 1$ for $b = a(2r+1)$, where $a \in \{0, \ldots, n-1\}$. Consequently for every $k \in \{0, 2, \ldots, 2r\}$

$$\mathrm{sgn}(f)|_{I_b} = \mathrm{sgn}(f)|_{(0,\frac{2\pi}{(2r+1)n})} = 1 \quad \text{for} \quad b = a(2r+1)+k,$$

where $a \in \{0, \ldots, n-1\}$ and for $k \in \{1, 3, \ldots, 2r-1\}$

$$\mathrm{sgn}(f)|_{I_b} = \mathrm{sgn}(f)|_{(0,\frac{2\pi}{(2r+1)n})} = -1 \quad \text{for} \quad b = a(2r+1)+k,$$

where $a \in \{0, \ldots, n-1\}$. For simplicity of notation let $I_{a,l} = I_{a(2r+1)+l}$ for $a \in \{0, 1, \ldots, n-1\}$ and $l \in \{0, 1, \ldots 2r\}$. Now, fix $k \in \mathbb{N} \setminus \{0\}$

$$\begin{aligned}
\int_0^{2\pi}\mathrm{sgn}(V_n(t))e^{ikt}dt &= \sum_{l=0}^{2r}(-1)^l\sum_{a=0}^{n-1}\int_{I_{a,l}}e^{ikt}dt \\
&= (ik)^{-1}\sum_{l=0}^{2r}(-1)^l\sum_{a=0}^{n-1}\left(e^{\frac{2\pi ik(a(2r+1)+l+1)}{(2r+1)n}} - e^{\frac{2\pi ik(a(2r+1)+l)}{(2r+1)n}}\right) \\
&= (ik)^{-1}\sum_{l=0}^{2r}(-1)^l e^{\frac{2\pi ik(l+1)}{(2r+1)n}}\sum_{a=0}^{n-1}\left(e^{\frac{2\pi ik}{n}}\right)^a \\
&\quad - (ik)^{-1}\sum_{l=0}^{2r}(-1)^l e^{\frac{2\pi ikl}{(2r+1)n}}\sum_{a=0}^{n-1}\left(e^{\frac{2\pi ik}{n}}\right)^a.
\end{aligned}$$

If $rn < |k| < (r+1)n$, then $e^{\frac{2\pi ik}{n}} \neq 1$. Hence

$$\int_0^{2\pi} \operatorname{sgn}(V_n(t))e^{ikt}dt = (ik)^{-1}\sum_{l=0}^{2r}(-1)^l\big(e^{\frac{2\pi ik(l+1)}{(2r+1)n}} - e^{\frac{2\pi ikl}{(2r+1)n}}\big)\frac{1-e^{2\pi ik}}{1-e^{\frac{2\pi ik}{n}}} = 0,$$

which yields the conclusion. □
By *Theorem 7* we have the following formula for the norm of the operator H^1_{rn}.

Theorem 13 *Let $X = \mathscr{C}_{2\pi}(\mathbb{R})$ or $X = L_1[0,2\pi]$. Fix $r \in \mathbb{N}$. Then for all $n \in \mathbb{N}$*

$$\|H^1_{rn}\| = L_r, \tag{24}$$

where L_r is the r-th Lebesgue constant ($L_r = \|F_r\| = \|D_r\|$).

Theorem 14 *Let $X = \mathscr{C}_{2\pi}(\mathbb{R})$ or $X = L_1[0,2\pi]$. Then for all $r \in \mathbb{N}$, the operator H^2_{rn} is MGP in $\mathscr{P}_{\Pi_{rn}}(X, \Pi_{(r+2)n-1})$.*

PROOF. By *Lemma 10* it is sufficient to show that for every $k \in \mathbb{Z}$, such that $|k| \in (rn,(r+2)n)$ holds $\int_0^{2\pi}\operatorname{sgn}(V^2_{rn}(t))e^{ikt}dt = 0$. Accordingly to *Lemma 11* $V^2_{rn}(t) = 0$ iff $t = \frac{\pi b}{(r+1)n}$ for $b \in \{1,\dots,2(r+1)n-1\}$. Moreover only in double zeros of V^2_{rn} ($t = \frac{\pi a}{n}$ for $a \in \{1,\dots 2n-1\}$) does not change the sign. Denote $I_b := \left(\frac{\pi b}{(r+1)n}, \frac{\pi(b+1)}{(r+1)n}\right)$ for $b \in \{0,\dots,2(r+1)n-1\}$. Observe that between two consecutive double zeros there are exactly r simple zeros. Hence $\operatorname{sgn}(f)|_{I_b} = \operatorname{sgn}(f)|_{I_{b+2l(r+1)}}$ for all $b \in 0,\dots,2r+1$ and $l \in \{1,\dots,n-1\}$. Fix $k \in \mathbb{N}\setminus\{0\}$, then

$$\begin{aligned}\int_0^{2\pi}\operatorname{sgn}(V^2_{rn}(t))e^{ikt}dt &= \sum_{b=0}^{2r+1}\operatorname{sgn}(f)|_{I_b}\sum_{l=0}^{n-1}\int_{I_{b+2l(r+1)}} e^{ikt}dt\\ &= (ik)^{-1}\sum_{b=0}^{2r+1}\operatorname{sgn}(f)|_{I_b}\sum_{l=0}^{n-1}\big(e^{\frac{\pi ik(b+1+2l(r+1))}{(r+1)n}} - e^{\frac{\pi ik(b+2l(r+1))}{(r+1)n}}\big)\\ &= (ik)^{-1}\sum_{b=0}^{2r+1}\operatorname{sgn}(f)|_{I_b}\big(e^{\frac{\pi ik(b+1)}{(r+1)n}} - e^{\frac{\pi ikb}{(r+1)n}}\big)\sum_{l=0}^{n-1}(e^{\frac{2\pi ik}{n}})^l\end{aligned}$$

If $rn < |k| < (r+1)n$ or $(r+1)n < |k| < (r+2)n$ then $e^{\frac{2\pi ik}{n}} \neq 1$. Hence

$$\int_0^{2\pi}\operatorname{sgn}(V^2_{rn}(t))e^{ikt}dt = (ik)^{-1}\sum_{b=0}^{2r+1}\operatorname{sgn}(f)|_{I_b}\big(e^{\frac{\pi ik(b+1)}{(r+1)n}} - e^{\frac{\pi ikb}{(r+1)n}}\big)\frac{1-e^{2\pi ik}}{1-e^{\frac{2\pi ik}{n}}} = 0.$$

On the other hand for $|k| = (r+1)n$, we have

$$\begin{aligned}\int_0^{2\pi}\operatorname{sgn}(V^2_{rn}(t))e^{ikt}dt &= (ik)^{-1}\sum_{b=0}^{2r+1}2n\operatorname{sgn}(f)|_{I_b}(-1)^{b+1}\\ &= (ik)^{-1}\sum_{b=0}^{r}2n(-1)^{2b+1} + (ik)^{-1}\sum_{b=r+1}^{2r+1}2n(-1)^{2b} = 0,\end{aligned}$$

which completes the proof. □
By the above calculations it is easy to deduce the formula for $\|H^2_{rn}\|$.

Theorem 15 *Let $X = \mathscr{C}_{2\pi}(\mathbb{R})$ or $X = L_1[0,2\pi]$. Then for all odd $r \in \mathbb{N}$*

$$\|H_{rn}^2\| = \frac{4}{\pi}\sum_{m=0}^{\frac{r-1}{2}} \frac{\tan\left(\frac{(2m+1)\pi}{2(r+1)}\right)}{2m+1}. \tag{25}$$

PROOF. Using *Theorem 7* we get $\|H_{rn}^2\| = \|H_r^2\|$. Accordingly by *Definition 6* we have

$$\begin{aligned}\|H_r^2\| &= \frac{1}{2\pi}\int_0^{2\pi} |V_r^2(t)|\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \operatorname{sgn}(V_r^2(t))\Big(\sum_{k=-r}^{r} e^{ikt} + \frac{1}{2}(e^{i(r+1)t} + e^{-i(r+1)t})\Big)\,dt.\end{aligned}$$

By the proof of *Theorem 14*

$$\int_0^{2\pi} \operatorname{sgn}(V_r^2(t))(e^{i(r+1)t} + e^{-i(r+1)t})\,dt = 0,$$

and for $k \in \mathbb{Z}$ such that $0 < |k| \le r$

$$\begin{aligned}a_k &:= \int_0^{2\pi} \operatorname{sgn}(V_r^2(t))e^{ikt}\,dt = (ik)^{-1}\sum_{b=0}^{r}(-1)^b(e^{\frac{\pi ik(b+1)}{r+1}} - e^{\frac{\pi ikb}{r+1}}) \\ &+ (ik)^{-1}\sum_{b=r+1}^{2r+1}(-1)^{b-1}(e^{\frac{\pi ik(b+1)}{r+1}} - e^{\frac{\pi ikb}{r+1}}) = (ik)^{-1}\sum_{b=0}^{r}(-1)^b(e^{\frac{\pi ik(b+1)}{r+1}} - e^{\frac{\pi ikb}{r+1}}) \\ &+ \frac{e^{\pi ik}}{ik}\sum_{b=0}^{r}(-1)^{b+r}(e^{\frac{\pi ik(b+1)}{r+1}} - e^{\frac{\pi ikb}{r+1}}).\end{aligned}$$

Hence for $k = 2m$, $a_k = 0$, and for $k = 2m+1$ we have

$$\begin{aligned}a_k &= \frac{2}{ik}\sum_{b=0}^{r}(-1)^b(e^{\frac{\pi ik(b+1)}{r+1}} - e^{\frac{\pi ikb}{r+1}}) = 4\left(\sum_{k=0}^{\frac{r-1}{2}} e^{\frac{\pi ik(2b+1)}{r+1}} - \sum_{k=1}^{\frac{r-1}{2}} e^{\frac{2\pi ikb}{r+1}}\right) \\ &= 4(e^{\frac{\pi ik}{r+1}} - 1)\sum_{k=1}^{\frac{r-1}{2}}\left(e^{\frac{2\pi ikb}{r+1}} + 4e^{\frac{\pi ik}{r+1}}\right) = \frac{4}{ik}\cdot\frac{e^{\frac{\pi ik}{2(r+1)}} - e^{\frac{-\pi ik}{2(r+1)}}}{e^{\frac{\pi ik}{2(r+1)}} + e^{\frac{-\pi ik}{2(r+1)}}} = \frac{4}{k}\tan\big(\frac{\pi k}{2(r+1)}\big).\end{aligned}$$

Notice that zeros of the function V_r^2 divide the interval $(0,\pi)$ into $r+1$ equal parts. Since all zeros in this interval are simple, sign of the function is equal 1 on $\frac{r+1}{2}$ parts and -1 on the $\frac{r+1}{2}$ parts. Hence by *Lemma 11* (2) $a_0 = 0$. Now we can easily compute the norm

$$\|H_{rn}^2\| = \|H_r^2\| = \frac{1}{2\pi}\sum_{k=-r}^{k=r} a_k = \frac{1}{\pi}\sum_{m=0}^{\frac{r-1}{2}} a_{2m+1} = \frac{4}{\pi}\sum_{m=0}^{\frac{r-1}{2}} \frac{\tan\left(\frac{(2m+1)\pi}{2(r+1)}\right)}{2m+1}. \quad \square$$

Now we present a theorem, which shows that de la Vallée Poussin's type operators do not have to be (MGP).

Theorem 16 *Let $n, s \in \mathbb{N}$ and $s > 3$. Then $H_{n,sn}$ is not a minimal generalized projection in $\mathscr{P}_{\Pi_n}(X, \Pi_{sn-1})$.*

PROOF. To the contrary, assume that $H_{n,sn}$ is MGP. According to *Lemma 10*, for all $|k| \in \{n+1, \ldots, sn-1\}$ we have $\int_0^{2\pi} \mathrm{sgn}(V_{n,sn})(t) e^{ikt}\, dt = 0$, and consequently

$$\int_0^{2\pi} \mathrm{sgn}(V_{n,sn}(t)) \cos(kt)\, dt = 0 \quad \text{for all } \; k \in \{n+1, \ldots, sn-1\}. \tag{26}$$

Since $s > 3$, $n < (s-2)n < sn$. The functions $\cos((s-2)nt)$ and $\mathrm{sgn}(V_{n,sn}(t))$ are $\frac{2\pi}{n}$ periodic. Hence by (26) we get

$$\int_0^{\frac{2\pi}{n}} \mathrm{sgn}(V_{n,sn}(t)) \cos((s-2)nt)\, dt = 0.$$

Using *Lemma 11* it is easy to see that $\mathrm{sgn}(V_{n,sn}(t)) = \mathrm{sgn}(V_{1,s}(nt))$ for $t \in (0, \frac{2\pi}{n})$, and consequently

$$\int_0^{2\pi} \mathrm{sgn}(V_{1,s}(t)) \cos((s-2)t)\, dt = 0.$$

Now assume that $s = 2l+1$ for some integer $l \geq 2$. In the proof of the next theorem we will show the equation ((29)) and thus

$$\int_0^{2\pi} \mathrm{sgn}(V_{1,s}(t)) \cos((s-2)t)\, dt = \frac{4}{s-2}\left(\cot\left(\frac{\pi(s-2)}{s+1}\right) - \cot\left(\frac{\pi(s-2)}{s-1}\right)\right) \neq 0,$$

which leads to the contrary. Now consider $s = 2l$ for some integer $l \geq 2$. Then by (31) below

$$\int_0^{2\pi} \mathrm{sgn}(V_{1,s}(t)) \cos((s-2)t)\, dt = \frac{2}{s-2}\left(\tan\left(\frac{\pi(s-2)}{2(s+1)}\right) - \tan\left(\frac{\pi(s-2)}{2(s-1)}\right)\right) \neq 0,$$

a contradiction. □

Theorem 17 *Let $n \geq 1$ and $l \geq 2$ be integers. Then*

$$\|H_{n,(2l+1)n}\| = \frac{4}{\pi} \sum_{m=0}^{l-1} \frac{l-m}{l(2m+1)} \left(\cot\left(\frac{(2m+1)\pi}{2(l+1)}\right) - \cot\left(\frac{(2m+1)\pi}{2l}\right)\right)$$

and

$$\|H_{n,2ln}\| = \frac{2}{\pi} \sum_{m=0}^{l-1} \frac{2(l-m)-1}{(2l-1)(2m+1)} \left(\cot\left(\frac{(2m+1)\pi}{2(2l+1)}\right) - \cot\left(\frac{(2m+1)\pi}{2(2l-1)}\right)\right)$$
$$+ \frac{2}{\pi} \sum_{m=1}^{l-1} \frac{l-m}{(2l-1)m} \left(\tan\left(\frac{m\pi}{2l-1}\right) - \tan\left(\frac{m\pi}{2l+1}\right)\right) + \frac{1}{4l^2-1}.$$

PROOF. Using *Theorem 7* we get $\|H_{n,sn}\| = \|H_{1,s}\|$. By *Definition 6*

$$\|H_{1,s}\| = \frac{1}{2\pi}\int_0^{2\pi} |V_{1,s}(t)|\,dt = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{sgn}(V_{1,s}(t))\Big(1+\sum_{k=1}^{s-1}\frac{s-k}{s-1}(e^{ikt}+e^{-ikt})\Big)dt.$$

For $k \in \mathbb{Z}$, denote

$$a_k := \int_0^{2\pi} \mathrm{sgn}(V_{1,s}(t))e^{ikt}\,dt.$$

Now, fix k such that $|k| \in \{1,\dots s-1\}$. Then by *Lemma 11*

$$\begin{aligned}
a_k &= \int_0^{\pi} \mathrm{sgn}(V_{1,s}(t))e^{ikt}\,dt + \int_0^{\pi} \mathrm{sgn}(V_{1,s}(2\pi-t))e^{ik(2\pi-t)}\,dt \\
&= \int_0^{\pi} \mathrm{sgn}(V_{1,s}(t))(e^{ikt}+e^{-ikt})\,dt = 2\sum_{a=1}^{\left[\frac{s}{2}\right]}\int_{\frac{2\pi(a-1)}{s-1}}^{\frac{2\pi a}{s+1}}\cos(kt)\,dt \\
&- 2\sum_{a=1}^{\left[\frac{s}{2}\right]-1}\int_{\frac{2\pi a}{s+1}}^{\frac{2\pi a}{s-1}}\cos(kt)\,dt - 2\int_{\frac{2\pi\left[\frac{s}{2}\right]}{s+1}}^{\pi}\cos(kt)\,dt \\
&= \frac{4}{k}\sum_{a=1}^{\left[\frac{s}{2}\right]}\sin\left(\frac{2\pi k}{s+1}a\right) - \frac{4}{k}\sum_{a=1}^{\left[\frac{s}{2}\right]-1}\sin\left(\frac{2\pi k}{s-1}a\right).
\end{aligned}$$

By the above computation it is easy to see that $a_{-k} = a_k$, for all $k \in \{1,\dots,s-1\}$. Hence

$$\|H_{1,s}\| = \frac{1}{2\pi}(a_0 + 2\sum_{k=1}^{s-1}a_k). \tag{27}$$

Now assume that $s = 2l+1$ for some $l \in \mathbb{N}_2$. In this case π is a double zero of the function $\mathrm{sgn}(V_{1,s})$, so by *Lemma 11* (2) $a_0 = 0$. If $k = 2m$ for some $m \in \{1,\dots,l-1\}$, then by the above and the following trigonometric identity

$$\sum_{k=1}^{n}\sin kt = \frac{\sin\left(\frac{n+1}{2}t\right)\sin\left(\frac{n}{2}t\right)}{\sin\left(\frac{t}{2}\right)}. \tag{28}$$

we obtain

$$a_k = \frac{2}{m}\left(\frac{\sin(\pi m)\sin\left(\frac{\pi m l}{l+1}\right)}{\sin\left(\frac{\pi m}{l+1}\right)} - \frac{\sin(\pi m)\sin\left(\frac{\pi m(l-1)}{l}\right)}{\sin\left(\frac{\pi m}{l}\right)}\right) = 0,$$

moreover for $k = 2l$ we have

$$a_k = \frac{4}{k}\sum_{a=1}^{l}\sin\left(\frac{2\pi l}{l+1}a\right) = \frac{2\sin(\pi l)\sin\left(\frac{\pi l^2}{l+1}\right)}{l\sin\left(\frac{\pi l}{l+1}\right)} = 0$$

Analogously for $k = 2m+1$ ($m \in \{0,\dots,l-1\}$) we can compute

$$\begin{aligned} a_k &= \frac{4}{k}\left(\frac{\sin(\frac{\pi k}{2})\sin\left(\frac{\pi k}{2}-\frac{\pi k}{2(l+1)}\right)}{\sin(\frac{\pi k}{2(l+1)})} - \frac{\sin(\frac{\pi k}{2})\sin\left(\frac{\pi k}{2}-\frac{\pi k}{2l}\right)}{\sin\left(\frac{\pi k}{2l}\right)}\right) \\ &= \frac{4}{k}\left(\cot\left(\frac{\pi k}{2(l+1)}\right) - \cot\left(\frac{\pi k}{2l}\right)\right). \end{aligned} \tag{29}$$

Hence by (27)

$$\|H_{1,2m+1}\| = \frac{4}{\pi}\sum_{m=0}^{l-1}\frac{l-m}{l(2m+1)}\left(\cot\left(\frac{(2m+1)\pi}{2(l+1)}\right) - \cot\left(\frac{(2m+1)\pi}{2l}\right)\right),$$

which completes the proof of the first part of our theorem.
Now we are taking $s = 2l$ for some integer $l \geq 2$. Then

$$a_0 = 4\sum_{a=1}^{l}\frac{2\pi a}{(s+1)} - 4\sum_{a=1}^{l-1}\frac{2\pi a}{(s-1)} - 2\pi = \frac{2\pi}{4l^2-1}, \tag{30}$$

and for $k = 2m$, where $m \in \{1,\dots,l-1\}$

$$\begin{aligned} a_k = \frac{4}{k}&\left(\frac{\sin(\pi m + \frac{\pi m}{2l+1})\sin(\pi m - \frac{\pi m}{2l+1})}{\sin(2\frac{\pi m}{2l+1})}\right. \\ &\left. - \frac{\sin(\pi m + \frac{\pi m}{2l-1})\sin(\pi m - \frac{\pi m}{2l-1})}{\sin(2\frac{\pi m}{2l-1})}\right) \\ = \frac{2}{k}&\left(\tan\left(\frac{\pi m}{2l-1}\right) - \tan\left(\frac{\pi m}{2l+1}\right)\right). \end{aligned} \tag{31}$$

Moreover for $k = 2m+1$, where $m \in \{0,\dots,l-1\}$

$$\begin{aligned} a_k = \frac{4}{k}&\left(\frac{\sin\left(\frac{\pi(2m+1)}{2}+\frac{\pi(2m+1)}{2(2l+1)}\right)\sin\left(\frac{\pi(2m+1)}{2}-\frac{\pi(2m+1)}{2(2l+1)}\right)}{\sin\left(2\frac{\pi(2m+1)}{2(2l+1)}\right)}\right. \\ &\left. - \frac{\sin\left(\frac{\pi(2m+1)}{2}+\frac{\pi(2m+1)}{2(2l-1)}\right)\sin\left(\frac{\pi(2m+1)}{2}-\frac{\pi(2m+1)}{2(2l-1)}\right)}{\sin\left(2\frac{\pi(2m+1)}{2(2l-1)}\right)}\right) \\ = \frac{2}{k}&\left(\cot\left(\frac{\pi(2m+1)}{2(2l+1)}\right) - \cot\left(\frac{\pi(2m+1)}{2(2l-1)}\right)\right). \end{aligned} \tag{32}$$

Hence by (27)

$$\begin{aligned} \|H_{1,2l}\| = &\frac{2}{\pi}\sum_{m=0}^{l-1}\frac{2(l-m)-1}{(2l-1)(2m+1)}\left(\cot\left(\frac{(2m+1)\pi}{2(2l+1)}\right) - \cot\left(\frac{(2m+1)\pi}{2(2l-1)}\right)\right) \\ &+ \frac{2}{\pi}\sum_{m=1}^{l-1}\frac{l-m}{(2l-1)m}\left(\tan\left(\frac{m\pi}{2l-1}\right) - \tan\left(\frac{m\pi}{2l+1}\right)\right) + \frac{1}{4l^2-1}, \end{aligned}$$

as required. □

Theorem 18 *For all $n \in \mathbb{N}$ we have*

$$\|H_{n,sn}\| = 1 + \Theta\left(\frac{1}{s}\right). \tag{33}$$

PROOF. Observe first that the upper estimation is a straightforward consequence of the Cauchy-Schwarz inequality. Indeed

$$\begin{aligned}\|H_{rn,sn}\| &= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{|\sin x\sin(\frac{s+r}{s-r}x)|}{x^2}dx = \frac{1}{\pi}\int_{-\infty}^{+\infty}\left|\frac{\sin x}{x}\cdot\frac{\sin(\frac{s+r}{s-r}x)}{x}\right|dx\\ &\le \frac{1}{\pi}\left(\int_{-\infty}^{+\infty}\frac{\sin^2 x}{x^2}dx\right)^{1/2}\left(\int_{-\infty}^{+\infty}\frac{\sin^2(\frac{s+r}{s-r}x)}{x^2}dx\right)^{1/2}\\ &= \frac{1}{\pi}\cdot\pi^{1/2}\cdot\sqrt{\frac{s+r}{s-r}}\cdot\pi^{1/2} = \sqrt{1+\frac{2r}{s-r}} \le 1+\frac{r}{s-r}.\end{aligned}$$

The main difficulty in Theorem 18 is in the lower estimation.
By *Theorem 7*, (18) and (19) we get

$$\|H_{n,sn}\| = \|H_{1,s}\| = \frac{1}{2\pi(s-1)}\int_0^{2\pi}\left|\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}-1\right|dt.$$

Let Ω be a subset of $[0,2\pi]$ and $\widehat{\Omega} := [0,2\pi]\setminus\Omega$. Then

$$\begin{aligned}\|H_{n,sn}\| &\ge \frac{1}{2\pi(s-1)}\left(\int_{\widehat{\Omega}}\left(\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}-1\right)dt + \int_{\Omega}\left(1-\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}\right)dt\right)\\ &= \frac{1}{2\pi(s-1)}\left(\int_0^{2\pi}\left(\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}-1\right)dt + 2\int_{\Omega}\left(1-\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}\right)dt\right)\\ &= \frac{1}{2\pi(s-1)}\left(2\pi s-2\pi+2\int_{\Omega}\left(1-\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}\right)dt\right)\\ &= 1+\frac{1}{\pi(s-1)}\int_{\Omega}\left(1-\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}\right)dt. \end{aligned} \tag{34}$$

Observe that

$$K(t) = 1-\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})} = \frac{\cos(st)-\cos(t)}{1-\cos(t)}. \tag{35}$$

Let $\Omega = \bigcup_k [\frac{2\pi}{s}k-\frac{c}{s}, \frac{2\pi}{s}k+\frac{c}{s}]$ where the summation is over all integers k such that $\frac{2\pi}{s}k$ is in the interval $[\frac{\pi}{2}, \frac{3\pi}{2}]$ and c is a fixed number $0 < c \le \frac{\pi}{4}\cdot\frac{s}{s+1}$. Note that on Ω it holds $K(t) \ge 0$ and as a result $K(t) \ge \frac{1}{2}(\cos(st)-\cos(t))$. As a result, using (34), we have

$$\|H_{n,sn}\| \ge 1+\frac{1}{\pi(s-1)}\int_{\Omega}\left(1-\frac{\sin^2(\frac{st}{2})}{\sin^2(\frac{t}{2})}\right)dt \ge 1+\frac{1}{2\pi(s-1)}\int_{\Omega}\cos(st)-\cos(t)dt$$

Since $\Omega \subset [\frac{\pi}{2} - \frac{c}{s}, \frac{3\pi}{2} + \frac{c}{s}]$ then $\int_\Omega (-\cos(t))dt > 0$. Moreover

$$\int_{\frac{2\pi}{s}k-\frac{c}{s}}^{\frac{2\pi}{s}k+\frac{c}{s}} \cos(st)dt = \frac{2\sin(c)}{s}. \tag{36}$$

Therefore

$$\begin{aligned}\|H_{n,sn}\| &\geq 1 + \frac{1}{2\pi(s-1)} \int_\Omega \cos(st)dt \geq 1 + \frac{1}{2\pi(s-1)} \cdot \frac{2\sin(c)}{s} \cdot \frac{s-1}{2} \\ &\geq 1 + \frac{0.11}{s}. \end{aligned}\tag{37}$$

We can asymptotically improve on the above estimate as follows

$$\begin{aligned}\int_\Omega (-\cos(t))dt = \sum_k \int_{\frac{2\pi}{s}k-\frac{c}{s}}^{\frac{2\pi}{s}k+\frac{c}{s}} -\cos(t)dt &= \sum_k \sin(\frac{2\pi}{s}k - \frac{c}{s}) - \sin(\frac{2\pi}{s}k + \frac{c}{s}) \\ &= \sum_k -2\sin(\frac{c}{s})\cos(\frac{2\pi}{s}k) \\ &= 2\sin(\frac{c}{s})(\sum_k -\cos(\frac{2\pi}{s}k)) \end{aligned}\tag{38}$$

Note that $\sum_k(-\cos(\frac{2\pi}{s}k)) \cdot \frac{2\pi}{s}$ is a Riemann sum of $\int_{\pi/2}^{3\pi/2}(-\cos(t)dt) = 2$. Therefore $\sum_k(-\cos(\frac{2\pi}{s}k)) \approx 2 \cdot \frac{s}{2\pi} = \frac{s}{\pi}$. Hence

$$\int_\Omega (-\cos(t))dt \approx \frac{2c}{\pi}. \tag{39}$$

Combining this and (36) we get

$$\int_\Omega (\cos(st) - \cos(t))dt \approx \sin(c) + \frac{2c}{\pi}.$$

Remark 19 *Letting $s \to \infty$ and setting $c = \frac{\pi}{4}$ we obtain*

$$1 + \frac{0.192}{s-1} < \|H_{n,sn}\| < 1 + \frac{1}{s-1},$$

for s sufficiently large.

By the above theorem and *Corollary 8* we immediately get the following result.

Corollary 20 *Let $n \in \mathbb{N}$. Then*

$$\int_{-\infty}^{+\infty} \frac{|\sin x \sin(1+\frac{1}{n})x|}{x^2} dx = \pi + \Theta\left(\frac{1}{n}\right)$$

3 Rademacher projections

We contrast the results obtained for the trigonometric system to the results obtained for the Rademacher system. The well-known Rademacher functions, $r_0, r_1, \ldots$, defined by $r_j(t) = (-1)^{[2^j t]}$ for $0 \le t \le 1$ plays a central role in many areas of analysis ($[x]$ denotes the integer part of x). For further investigations we will need a notion of **dyadic group**. We shall denote the set of **dyadic rationals** in the unit interval $[0,1)$ by Q. In particular, each element of Q has a form $p/2^n$ for some $p,n \in \mathbb{N}, 0 \le p < 2^n$. Any $x \in [0,1]$ may be written in the form

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}, \tag{40}$$

where each $x_k = 0$ or 1. For each $x \in [0,1] \setminus Q$ there is only one expression of this form. We shall call it the **dyadic expansion** of x. When $x \in Q \setminus \{0\}$ there are two expression of this form, one which terminates in 0's and one which terminates in 1's. By the dyadic expansion of $x \in Q$ we shall mean the one which terminates in 0's. Notice that $1 \notin Q$ so the dyadic expansion of $x = 1$ terminates in 1's.
Now we can define the **dyadic addition** of two numbers x, y by

$$x \oplus y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}. \tag{41}$$

Observe that $x \oplus x = 0$ therefore $x \oplus y = x \ominus y$.

Theorem 21 (see e.g. [17]) *The following holds true*

$$r_n(x \oplus y) = r_n(x) r_n(y) \qquad \textit{for } x \oplus y \notin Q \tag{42}$$

For many other interesting facts concerning Rademacher functions the reader is referred to [17]. Let A be a nonempty finite subset of $\mathbb{N}$. Denote

$$Rad_A = span\{r_k : \ k \in A\}. \tag{43}$$

The Rademacher projection is defined by

$$R_A = \sum_{i \in A} r_i \otimes r_i : L_1[0,1] \to Rad_A. \tag{44}$$

We can write the above projection as

$$R_A(f) = \sum_{i \in A} \left(\int_0^1 r_i(t) f(t)\, dt \right) r_i, \tag{45}$$

or using Dirichlet kernel $D_A^r = \sum_{i \in A} r_i$ and orthogonality of Rademacher functions as

$$(R_A f)(s) = \int_0^1 f(t) D_A^r(t \oplus s)\, dt. \tag{46}$$

Lemma 22 *Let $A \subset B$ be nonempty finite subsets of $\mathbb{N}$. The norm of Rademacher projection R_A restricted to the subspace Rad_B is equal* 1.

PROOF. Let $Y := \text{span}\{r_k : \ k \in B \setminus A\} \subset L_1[0,1]$. Since for every function $f := \sum_{l \in B} a_l r_l \in Rad_B$ we have $R_A f = \sum_{l \in A} a_l r_l$, it is sufficient to show that 0 is a best approximation to $R_A f$ in Y. According to *Theorem 5* it suffices to prove that for every $k \in B \setminus A$ we have $\int_0^1 \text{sgn}(R_A f(s)) r_k(s)\, ds = 0$. Fix $k \in B \setminus A$. Denote

$$I_{j,k} := \left[\frac{j-1}{2^k}, \frac{j}{2^k}\right) \quad \text{for} \quad j \in \{1, \dots, 2^k\}.$$

Observe that, if $l < k$ then

$$r_l|_{I_{2m-1,k} \cup I_{2m,k}} \equiv const \quad \text{for} \quad m \in \{1, \dots, 2^{k-1}\}.$$

Moreover for $l > k$ we have

$$r_l\left(t + \frac{1}{2^k}\right) = r_l(t).$$

Hence if $t \in I_{2m-1,k}$ for some $m \in \{1, \dots, 2^{k-1}\}$ then

$$R_A f\left(t + \frac{1}{2^k}\right) = \sum_{l \in A} a_l r_l \left(t + \frac{1}{2^k}\right) = \sum_{l \in A} a_l r_l(t) = R_A f(t),$$

which implies

$$\int_0^1 \text{sgn}(R_A f(t)) r_k(t)\, ds = \sum_{j=1}^{2^k} \int_{I_{j,k}} \text{sgn}(R_A f(t)) (-1)^{j+1}\, ds = 0,$$

as required. □

Theorem 23 *Let $A \subset B$ be nonempty finite subsets of $\mathbb{N}$. Then the Rademacher projection R_A is a minimal generalized projection in $\mathscr{P}_{Rad_A}(L_1[0,1], Rad_B)$.*

PROOF. Fixed $Q \in \mathscr{P}_{Rad_A}(L_1[0,1], Rad_B)$. Then $\widetilde{Q} := R_A \circ Q$ is a projection from $L_1[0,1]$ onto Rad_A and by *Lemma 22* $\|\widetilde{Q}\| \le \|Q\|$, which implies

$$\lambda(Rad_A, L_1[0,1]) \le \lambda_{Rad_B}(Rad_A, L_1[0,1]).$$

Consequently, the Rademacher projection R_A, which is minimal in $\mathscr{P}(L_1[0,1], Rad_A)$, is also minimal generalized projection in $\mathscr{P}_{Rad_A}(L_1[0,1], Rad_B)$. □

By the symmetry of Rademacher spaces and the well-known estimation of the classical Rademacher projection norm based on Khinchin's inequality (see [6]) we have the following result

Theorem 24 (see e.g. [17]) *Let A be a nonempty finite subset of $\mathbb{N}$. Then*

$$\|R_A\| = \|R_n\| = \Theta(\sqrt{n}), \tag{47}$$

where n denotes the cardinality of A.

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