Overview of the Mathematics of Compressive Sensing

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"Compressive Sensing, Extensions, and Applications"
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Optimality of Uniform Guarantees

Gelfand Widths

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For a subset K of a normed space X, define

$$E^m(K,X) := \inf\left\{\sup_{\mathbf{x}\in K}\|\mathbf{x}-\Delta(A\mathbf{x})\|,\ A : X \overset{\text{linear}}{\rightarrow} \mathbb{R}^m, \Delta : \mathbb{R}^m \rightarrow X\right\}$$

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and if in addition $K + K \subseteq a\,K$, then

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$$\|\mathbf{x} - \Delta_1(A\mathbf{x})\|_p \leq \frac{C}{s^{1-1/p}} \sigma_s(\mathbf{x})_1 \leq \frac{C}{s^{1-1/p}} \approx \frac{C'}{(m/\ln(eN/m))^{1-1/p}}.$$

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This gives an upper bound for $E^m(B_1^N, \ell_p^N)$, and in turn

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There exists $n \geq \left(\frac{N}{4s}\right)^{\frac{s}{2}}$ *subsets* $S^1, \ldots, S^n$ *of size* s *such that*

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Taking the logarithm yields

$$m \ge \frac{s}{\ln 9} \ln\left(\frac{N}{4s}\right).$$