Inner products from matrices

Several of the problems assigned for homework involve determining when, for an $n \times n$ matrix A, $\langle \mathbf{x}, \mathbf{y} \rangle := \mathbf{y}^* A \mathbf{x}$ is an inner product on $\mathbb{R}^n$ or $\mathbb{C}^n$. (B^* is the conjugate transpose of B: $B^*_{j,k} = \bar{B}_{k,j}$. When B is real, it's the transpose.)

It is easy to show that to meet the requirements of symmetry/conjugate symmetry, homogeneity and additivity for $\langle \mathbf{x}, \mathbf{y} \rangle$ to be an inner product, the matrix A has to be Hermition – $A^* = A$. Positivity is usually the hard one to meet. It wil hold if and only if the eigenvalues of A are positive. I'm not going to give a proof, but just give a 2×2 exmples.

Example 0.1. Let $A = \begin{pmatrix} 13 & 5 \\ 5 & 13 \end{pmatrix}$. Since $A^* = A^T = A$, the matrix meets the condition of being Hermitian. The eigenvalues are the roots of $\lambda^2 - 26\lambda + 144 = 0$. Solving this we get $\lambda_1 = 16$ and $\lambda_2 = 8$. In addition, the eigenvectors are $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. These are orthogonal. We can normalize them so that they form an orthonormal set and put then put them into the matrix

$$S = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix},$$

which is orthogonal – i.e., $S^T = S^{-1}$. With a little bit of work, we can show that $A = S\Lambda S^T$, where $\Lambda = \begin{pmatrix} 16 & 0 \\ 0 & 8 \end{pmatrix}$. What does this mean for positivity? To get positivity, we must show that for $\mathbf{x} \neq \mathbf{0}$,

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^* A \mathbf{x} > 0$$

Using $A = S\Lambda S^T$, the inner product takes the form $\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^* S \Lambda S^T \mathbf{x}$. If we let $\mathbf{y} = S^T \mathbf{x}$, and use $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, we end up with

$$\langle \mathbf{x}, \mathbf{x} \rangle = 16y_1^2 + 8y_2^2 > 0,$$

provided at least y_1 or y_2 is not 0. If both were 0, then $\mathbf{y}$ would be $\mathbf{0}$. Since $\mathbf{y} = S^T \mathbf{x} = S^{-1} \mathbf{x}$, this would mean that $\mathbf{x} = S\mathbf{y} = \mathbf{0}$. But we have assumed that $\mathbf{x} \neq \mathbf{0}$. The proof in the general case, even with complex scalars, follows from being able to put A in the form $A = S\Lambda S^T$ in the real case, and $A = S\Lambda S^*$ in the complex case. A theorem from inear algebra shows this can always be done.