The Closed Range Theorem
by
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Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the bounded linear operators on $\mathcal{H}$ and the compact operators on $\mathcal{H}$, respectively. Our aim here is to prove that the range of an operator of the form $L = I - \lambda K$, where K is compact, is closed. Specifically, we wish to prove the following theorem.

Theorem 0.1 (Closed Range Theorem)**.** *If $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, then the range of the operator $L = I - \lambda K$ is closed.*

Proof. The proof will be carried out in two steps.

Step 1. When $N(L)$, the null space of L, satisfies $N(L) \neq \{0\}$, the solution to $Lf = g$ is not unique. To make it unique, we simply project out the null space. Since $\mathcal{H} = N(L) \oplus N(L)^{\perp}$ the effect is to make L a one-to-one operator mapping $N(L)^{\perp}$ to $R(L)$.

We will now show that if $f \in N(L)^{\perp}$, then there is a constant $c > 0$, independent of f, such that

$$\|Lf\| \geq c\|f\|. \tag{0.1}$$

If not, then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset N(L)^{\perp}$ such that $\|f_n\| = 1$ and $\|Lf_n\| \to 0$ as $n \to \infty$. Note that $Lf_n = f_n - \lambda K f_n$, so

$$f_n = \lambda K f_n + L f_n.$$

Since the f_n's are bounded and K is compact, we may choose a subsequence $\{f_{n_k}\}$ such that $\{Kf_{n_k}\}$ is convergent. Thus,

$$\tilde{f} := \lim_{k\to\infty} f_{n_k} = \lambda \lim_{k\to\infty} K f_{n_k} + \lim_{k\to\infty} L f_{n_k},$$

since both terms on the right are convergent. Now, on the one hand, L is bounded and therefore continuous; hence, $L\tilde{f} = \lim_{k\to\infty} Lf_{n_k} = 0$ and $\tilde{f} \in N(L)$. On the other hand, $N(L)^{\perp}$ being closed and $\{f_n\}_{n=1}^{\infty} \subset N(L)^{\perp}$ imply that $\tilde{f} \in N(L)^{\perp}$. It follows that that $\tilde{f}$ is orthogonal to itself and is thus 0. However, $1 = \lim_{n\to\infty} \|f_n\| = \lim_{k\to\infty} \|f_{n_k}\| = \|\tilde{f}\| = 0$. This is a contradiction.

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Step 2. We want to show that if there is sequence $\{g_n\} \subset R(L)$ such that $g_n \to g$, then $g = Lf$ for some $f \in \mathcal{H}$. To begin, note that the solution f_n to $g_n = Lf_n$ is not unique if $N(L) \neq \{0\}$. But, as in step 1, we can make a unique choice by requiring that f_n be in $N(L)^{\perp}$. With this being the case, (0.1) holds and $\|g_n - g_m\| = L(f_n - f_m)\| \geq c\|f_n - f_m\|$. Because the convergent sequence $\{g_n\}$ is Cauchy, this inequality also implies that $\{f_n\}$ is Cauchy. Thus, $\{f_n\}$ is convergent to some $f \in \mathcal{H}$. It follows that $g = \lim_{n\to\infty} Lf_n = Lf$, so $g \in R(L)$. □

The Closed Range Theorem allows us to apply the Fredholm alternative to equations of the form $u - \lambda Ku = f$. Thus, we have the following result, which applies, for example, to finite rank or Hilbert-Schmidt operators.

Corollary 0.2. *Let $K \in \mathcal{C}(\mathcal{H})$ and $\lambda \in \mathbb{C}$. The equation $u - \lambda Ku = f$ has a solution if and only if $f \in N(I - \bar{\lambda}K^*)^{\perp}$.*

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