Applied/Numerical Analysis Qualifying Exam

August 11, 2015

Cover Sheet – Applied Analysis Part

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Name____

Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 11, 2015

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Notation: $\mathcal{H}$ denotes a complex, separable Hilbert space, with inner product and norm given by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. $\mathcal{B}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ are, respectively, the set of bounded linear operators on $\mathcal{H}$ and the set of compact linear operators on $\mathcal{H}$.

Problem 1. This problem is aimed at proving the Riemann-Lebesgue Lemma: *If $f \in L^1[0,1]$, then $\lim_{\lambda\to\infty} \int_0^1 f(x)e^{i\lambda x}dx = 0$.*

(a) Show that if $p(x) = \sum_{k=0}^n a_k x^k$, then $\lim_{\lambda\to\infty} \int_0^1 p(x)e^{i\lambda x}dx = 0$.
(b) State the Weierstrass Approximation Theorem. Use it and part (a) to show that for $g \in C[0,1]$, $\lim_{\lambda\to\infty} \int_0^1 g(x)e^{i\lambda x}dx = 0$.
(c) Use (a), (b) and the density of $C[0,1]$ in L^1 to complete the proof.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Let $\phi \in \mathcal{D}$ and define $\phi_h(x) := (\phi(x+h) - 2\phi(x) + \phi(x-h))/h^2$. Show that, in the sense of $\mathcal{D}$, $\lim_{h\to 0} \phi_h = \phi''$.
(b) Let $T \in \mathcal{D}'$ and define $T_h = (T(x+h) - 2T(x) + T(x-h))/h^2$. Show that, in the sense of distributions, $\lim_{h\to 0} T_h = T''$.

Problem 3. Let both $K \in \mathcal{C}(\mathcal{H})$ and $L \in \mathcal{B}(\mathcal{H})$ be self adjoint.

(a) Show that $\|L\| = \sup_{\|u\|=1} |\langle Lu, u\rangle|$. (Hint: look at $\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle$.)
(b) Prove this: *Either $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

Problem 4. Let L be a (possibly unbounded) closed, densely defined linear operator with domain $D_L \subseteq \mathcal{H}$.

(a) Define these: the resolvent set, $\rho(L)$; the discrete spectrum, $\sigma_d(L)$; the continuous spectrum, $\sigma_c(L)$; and the residual spectrum, $\sigma_r(L)$.
(b) Show that L^*, the adjoint of L, is closed and densely defined.
(c) Show that if L is self-adjoint, then $\sigma_r(L) = \emptyset$.

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Cover Sheet – Numerical Analysis Part

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NUMERICAL ANALYSIS QUALIFIER

August 11, 2015

In the problems below $\mathbb{P}^j$ denotes the space of polynomials on $\mathbb{R}^2$ of degree at most j.

Problem 1. Let $T \subset \mathbb{R}^2$ be a triangle with vertices v_1, v_2, and v_3. Let $p_1 = (v_1+v_2+v_3)/3$, $p_2 = (2v_1+v_2)/3$, $p_3 = (2v_1+v_3)/3$, $p_4 = v_2$, $v_5 = (v_2+v_3)/2$, and $p_6 = v_3$. Given $q \in \mathbb{P}^2$, let $\sigma_i(q) = q(p_i)$.

1. Show that the triple $(T, \mathbb{P}^2, \Sigma)$ constitutes a finite element, where $\Sigma = \{\sigma_i\}_{i=1}^6$.
2. Write down the nodal basis function ϕ_1 corresponding to this finite element. That is, $\phi_1 \in \mathbb{P}^2$ should satisfy $\phi_1(p_1) = 1$ and $\phi_1(p_j) = 0$, $j \neq 1$.

Hint: You should use barycentric (area) coordinates to derive your solution.

Problem 2. For $f \in L^2(0,1)$, consider the following weak formulation: Seek $(u,v) \in \mathbb{V} := H_0^1(0,1) \times H_0^1(0,1)$ satisfying for all $(\phi,\psi) \in \mathbb{V}$

$$a((u,v);(\phi,\psi)) := \int_0^1 u'\phi' + \int_0^1 v'\psi' - \int_0^1 v\phi = \int_0^1 f\psi =: L(\psi). \tag{2.1}$$

1. What is the corresponding strong form satisfied by u (eliminate v)?
2. Show that for all $w \in H_0^1(0,1)$
$$\left(\int_0^1 w^2\right)^{1/2} \leq \left(\int_0^1 |w'|^2\right)^{1/2}.$$
3. Using Part (2) show that $a(\cdot;\cdot)$ coerces the natural norm on $\mathbb{V}$:
$$|||\phi,\psi||| := \left(\|\phi\|_{H^1(0,1)}^2 + \|\psi\|_{H^1(0,1)}^2\right)^{1/2}$$
and explicitly find the coercivity constant.
4. Let $\mathbb{V}_h$ be a finite dimensional subspace of $\mathbb{V}$. Explain why there is a unique $(u_h,v_h) \in \mathbb{V}_h$ satisfying for all $(\phi_h,\psi_h) \in \mathbb{V}_h$
$$a((u_h,v_h);(\phi_h,\psi_h)) = L(\psi_h).$$
5. Show that
$$|||u-u_h, v-v_h||| \leq C_1 \inf_{(\phi_h,\psi_h)\in\mathbb{V}_h} |||u-\phi_h, v-\phi_h|||$$
(find C_1 explicitly).
6. You may assume that $u,v \in H_0^1(0,1) \cap H^2(0,1)$. Propose a discrete space $\mathbb{V}_h$ such that
$$|||u-u_h, v-v_h||| \leq C_2 h(\|u\|_{H^2(0,1)} + \|v\|_{H^2(0,1)})$$
for a constant C_2 independent of h. Justify your suggestion.

Problem 3. For $\Omega = (0,1)^2$ and $u_0 \in L^2(\Omega)$, consider the parabolic problem:

$$\begin{aligned} u_t - \Delta u + (u_x + u_y) &= 0, && (x,t) \in \Omega \times (0,T], \\ u(x,t) &= 0, && x \in \partial\Omega, t \in (0,T], \\ u(x,0) &= u_0(x), && x \in \Omega. \end{aligned} \tag{3.1}$$

1. Using a finite element space $V_h \subset H_0^1(\Omega)$, derive a semi-discrete approximation to (3.1) having solution $u_h(t) \in V_h$. This approximation satisfies $u_h(0) = \pi_h u_0$ with π_h denoting the $L^2(\Omega)$-projection onto V_h.
2. Show that
$$\|u_h(t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^2(\Omega)}, \qquad t \in [0, T].$$
Hint: Recall the integration-by-parts formula $\int_\Omega u v_{x_i} \mathrm{d}x = \int_{\partial\Omega} u v \nu_i \mathrm{d}\sigma - \int_\Omega u_{x_i} v \mathrm{d}x$, $u, v \in H^1(\Omega)$, where ν_i is the i-th component of the outward unit normal on $\partial\Omega$.
3. Consider the initial value problem:
$$w' + \lambda w = 0, \qquad w(0) = w_0,$$
and the time stepping method with step size k:
$$\frac{w^{n+1} - w^n}{k} + \lambda(\theta w^{n+1} + (1-\theta) w^n) = 0.$$
Here θ is a parameter in [0,1] and $\lambda \in \mathbb{R}$ with $\lambda > 0$. Use this method to develop a fully discrete (θ dependent) approximation to (3.1) (Note: $\theta = 1$ and $\theta = 0$ correspond to, respectively, backward and forward Euler time stepping).
4. Let $U^n \in V_h$ be the resulting fully discrete approximation after n steps using $U^0 = \pi_h u_0$. Show that for $\theta \in [1/2, 1]$,
$$\|U^n\|_{L^2(\Omega)} \leq \|U^0\|_{L^2(\Omega)}.$$
Hint: Test with a discrete function that depends on θ.