APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER
Applied Analysis Part, 2 hours
August 8, 2018

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let H be the Sobolev space $H = H_0^1[0,1] := \{f \in L^2[0,1] : f' \in L^2[0,1] \text{ and } f(0) = f(1) = 0\}$, where $\langle f, g\rangle_H := \int_0^1 f'(x)g'(x)dx$. You may assume that H is a Hilbert space and that if $f \in H$, then $f \in C[0,1]$.

(a) State and prove the Riesz Representation Theorem.
(b) Show that if $f \in H$, then $\|f\|_{C[0,1]} \le \|f\|_H$.
(c) Let $f \in C[0,1]$ and let the "delta" functional $\delta_x(f) = f(x)$. Use parts (a) and (b) to show that there exists $G(x,\cdot) \in H$ such that $f(x) = \langle f, G(x,\cdot)\rangle_H$ for all $f \in H$. Verify that $G(x,y) = G(y,x)$. (Hint: Let $f(\cdot) = G(y,\cdot)$.)
(d) Let $X := \{x_1 < x_2 < \cdots < x_n\} \subset \mathbb{R}$. Define the $n \times n$ matrix A by $A_{j,k} := G(x_j, x_k)$. If A is invertible, show that there exists a unique $s(x) = \sum_{k=1}^n c_k G(x, x_k) \in H$ such that s interpolates f on X – i.e., $s(x_j) = f(x_j)$, $j = 1, \dots, n$.

Problem 2. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and $\mathcal{D}'$.
(b) Let $\phi \in \mathcal{D}$ and define $\phi_h(x) := (\phi(x) - \phi(x-h))/h$. Show that, in the sense of $\mathcal{D}$, $\lim_{h\to 0} \phi_h = \phi'$. (Hint: apply Taylor's formula to $\phi_h^{(n)} - \phi^{(n+1)}$.)
(c) Let $T \in \mathcal{D}'$ and define $T_h = (T(x+h) - T(x))/h$. Show that, in the sense of distributions, $\lim_{h\to 0} T_h = T'$.
(d) Use (c) to show that if $T(x) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0, \end{cases}$ then $T'(x) = -\delta(x)$.

Problem 3. Let $k(x,y) = x^4 y^{12}$ and consider the operator $Ku(x) = \int_0^1 k(x,y)u(y)dy$.

(a) Show that K is a Hilbert-Schmidt operator and that $\|K\|_{\text{op}} \le 1/15$.
(b) State the Fredholm Alternative for the operator $L = I - \lambda K$. Explain why it applies in this case. Find all values of λ such that $Lu = f$ has a unique solution for all $f \in L^2[0,1]$.
(c) Use a Neumann series to find the resolvent $(I - \lambda K)^{-1}$ for λ small. Sum the series to find the resolvent.

Problem 4. Consider the one dimensional heat equation, $u_t = u_{xx}$, with $u(x,0) = f(x)$, where $-\infty < x < \infty$ and $0 < t < \infty$. Use Fourier transforms to show that the solution $u(x,t)$ is given by

$$u(x,t) = \int_{\mathbb{R}} K(x-y,t)f(y)dy, \;\; K(x-y,t) = (4\pi t)^{-1/2} e^{-(x-y)^2/4t}.$$

You may use any consistent Fourier transform convention[1] and any of the standard Fourier transform properties. You are also given that $\int_{-\infty}^{\infty} e^{-u^2 \pm ivu} du = \sqrt{\pi} e^{-v^2/4}$.

[1] For example, $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx$ & $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\xi)e^{ix\xi}dx$, or $\widehat{f}(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{ix\xi}dx$ & $f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)e^{-ix\xi}dx$.