Combined Applied Analysis/Numerical Analysis Qualifier
Applied Analysis Part
August 10, 2023

Instructions: *Do any 3 of the 4 problems in this part of the exam. Show all of your work clearly. Please indicate which of the 4 problems you are skipping.*

Problem 1. Let $\mathcal{H}$ be a (separable) Hilbert space and let $\mathcal{C}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$.

(a) Consider $K \in \mathcal{C}(\mathcal{H})$. Show that if $\{\phi_n\}_{n=0}^{\infty}$ is an orthonormal set in $\mathcal{H}$, then $\lim_{n\to\infty} K\phi_n = 0$.

(b) Suppose that $K \in \mathcal{C}(\mathcal{H})$ is self adjoint. Let $\lambda \neq 0$ be an eigenvalue of K. Show that the corresponding eigenspace is finite dimensional.

(c) Given that $\|K\| = \sup_{\|u\|=1} |\langle Ku, u\rangle|$, show that either $\|K\|$ or $-\|K\|$ (or possibly both) is an eigenvalue of K.

(d) *Briefly* explain how (b) and (c) are used to develop the spectral theory of compact self adjoint operators. (Two sentences will suffice.)

Problem 2. Let $\mathcal{P}$ be the set of all polynomials.

(a) State and sketch a proof of the Weierstrass approximation theorem.

(b) let $\mathcal{H} = L^2_w[0,1]$, where the inner product is $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}w(x)dx$ and where $w \in C[0,1]$, $w(x) \geq c > 0$ on $[0,1]$. Show that $\mathcal{P}$ is dense in $L^2_w[0,1]$. (You may use the density of $C[0,1]$ in $L^2[0,1]$.)

(c) Let $\mathcal{U} := \{p_n\}_{n=0}^{\infty}$ be the orthonormal set of polynomials obtained from $\mathcal{P}$ via the Gram-Schmidt process. Show that $\mathcal{U}$ is a complete orthonormal set in $L^2_w[0,1]$.

Problem 3. Suppose that $Tu(x) := \int_{-\infty}^{\infty} e^{-|x-y|}u(y)dy$.

(a) Show that T is a bounded operator on $L^2(\mathbb{R})$.

(b) You are given that the set $\phi_j = \chi_{[j,j+1]}$ is an orthonormal basis for $L^2(\mathbb{R})$. Show that $\|T\phi_j\| = \|T\phi_0\|$.

(c) Is T compact? Prove your answer.

Problem 4. Consider the operator $Lu = -u''$ defined on functions in $L^2[0,\infty)$ having u'' in $L^2[0,\infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0,\infty) \,|\, u'' \in L^2[0,\infty) \text{ and } u'(0) = 0\}.$$

(a) Find the Green's function $G(x, y; \lambda)$ for $-G'' - \lambda G = \delta(x - y)$, with $G'(0, y; \lambda) = 0$ and $\lambda \in \mathbb{C} \setminus [0,\infty)$.

(b) Is G a Hilbert-Schmidt kernel? Prove your answer.