Applied Analysis/Numerical Analysis Qualifying Exam

August 6, 2019

Numerical Analysis Part, 2 hours

Name____

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem so that it becomes trivial.

Problem 1. Consider the boundary value problem: Find u such that

$$-\Delta u = f \text{ in } \Omega,\ \nabla u \cdot \mathbf{n} + u = 0 \text{ on } \Gamma, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\Gamma = \partial\Omega$ is the boundary of Ω, $\mathbf{n}$ is the outward-pointing unit normal on Γ, and $q \in \mathbb{R}$ and $f \in L_2(\Omega)$ are given.
(a) The problem (1) has weak form given by: Find $u \in \mathbb{V}$ such that

$$a(u,v) = L(v),\ \forall v \in \mathbb{V}. \tag{2}$$

Identify the bilinear form a, the linear form L, and the function space $\mathbb{V}$.
(b) Show that the problem (2) has a unique solution.
Hint: If you have correctly identified $\mathbb{V}$, then there holds

$$\|u\|_{L_2(\Omega)} \le C(\|\nabla u\|_{L_2(\Omega)} + \|u\|_{L_2(\Gamma)}),\ u \in \mathbb{V}.$$

You may use this inequality without proof.
(c) Let $\mathcal{T}_h$ be a shape-regular partition of Ω into triangles. Introduce the finite dimensional space $\mathbb{V}_h$ consisting of continuous piecewise linear polynomials over $\mathcal{T}_h$. Consider the finite element approximation of (2): find

$$u_h \in \mathbb{V}_h, \quad \text{s.t.} \quad a(u_h, v) = L(v) \quad \text{for all} \quad v \in \mathbb{V}_h. \tag{3}$$

State and prove the optimal estimate for the error $\|u - u_h\|_{\mathbb{V}}$ assuming that the solution to (2) belongs to the Sobolev space $H^2(\Omega)$. As part of your proof you should define an appropriate interpolation operator and state, but not prove, optimal error estimates for this operator.
(d) Derive an optimal error bound for $\|u - u_h\|_{L^2(\Omega)}$ under the assumption of full regularity of the problem (2).

Problem 2. Consider the interval $I(0,1)$ and the set of continuous functions $\hat{v}$ defined on $[0,1]$. Let $\hat{a}_1 = 0$, $\hat{a}_2 = 1/4$, and $\hat{a}_3 = 1$. Consider also the following set of degrees of freedom:

$$\Sigma = \{\hat{v}(\hat{a}_1),\ \hat{v}(\hat{a}_3),\ \hat{v}'(\hat{a}_2)\}.$$

(a) Show that triple $(I, \mathbb{P}_2, \Sigma)$ is a finite element.
(b) Write down the basis for the quadratic polynomials $\mathbb{P}_2$ that is dual to Σ, that is, find $q_i \in \mathbb{P}_2$ $(i = 1,2,3)$ such that $\hat{q}_i(\hat{a}_j) = \delta_{ij}$ $(i = 1,2,3$ and $j = 1,3)$ and $\hat{q}_i'(\hat{a}_2) = \delta_{i2}$ $(i = 1,2,3)$. Then write down the finite element interpolant $\hat{\Pi}(\hat{w})$ of a given function $\hat{w} \in C^0[0,1]$ with respect to the given degrees of freedom.
(c) Consider the interval $[a,b]$, let F map $[0,1]$ onto $[a,b]$, and let $v \in H^3(a,b)$. Define $\Pi(v)$ by $(\Pi(v)) \circ F = \hat{\Pi}(v \circ F)$. Use the Bramble-Hilbert Lemma and the reference map F in order to estimate the error

$$\|v' - \Pi(v)'\|_{L_2(a,b)}$$

in terms of $h = b - a$. Explain how to modify the proof when v is less regular, in particular when $v \in H^2(a,b)$.

Problem 3. Let Ω be a bounded domain and $T > 0$ be a given final time. For $f \in C^0([0,T]; L_2(\Omega))$ and $u_0 \in H^1_0(\Omega)$ given, we consider the parabolic problem consisting in finding $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\begin{cases} \frac{\partial}{\partial t} u(x,t) - \Delta u(x,t) = f(x,t) & \text{for } (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

We assume that the solution u to the above problem is sufficiently smooth.

Let N be a strictly positive integer and let $\tau := T/N$, $t_n := n\tau$ and $t^{n+\frac{1}{2}} := \frac{1}{2}(t^{n+1} + t^n)$ for $n = 0, ..., N$. We consider the following semi-discretization in time: Set $U^0 := u_0$ and define $U^n : \Omega \to \mathbb{R}$ recursively by

$$\begin{cases} \frac{1}{\tau}(U^{n+1}(x) - U^n(x)) - \frac{1}{2}\Delta(U^{n+1}(x) + U^n(x)) = f(x, t^{n+\frac{1}{2}}) & \text{for } x \in \Omega, \\ U^{n+1}(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

(1) (Stability) Show that for $n = 0, ..., N$, U^n satisfies

$$\|U^{n+1}\|^2_{L_2(\Omega)} \leq \|U^0\|^2_{L_2(\Omega)} + \frac{1}{2} C_p^2 \tau \sum_{j=0}^{n} \|f(t^{j+\frac{1}{2}})\|^2_{L_2(\Omega)}.$$

(2) (Consistency I) Show either (but not both) that

$$\|\frac{1}{\tau}(u(t^{n+1}) - u(t^n)) - \frac{\partial}{\partial t} u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \leq C\tau^{\frac{3}{2}} \|\frac{\partial^3}{\partial t^3} u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))}$$

or

$$\|\frac{1}{2}\Delta\left(u(t^{n+1}) + u(t^n)\right) - \Delta u(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \leq C\tau^{\frac{3}{2}} \|\frac{\partial^2}{\partial t^2} \Delta u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))}.$$

Here C is a constant independent of τ, T and u.

Hint: You can use without proof the following Taylor expansion formula

$$g(b) = g(a) + g'(a)(b-a) + ... + \frac{1}{n!} g^{(n)}(a)(b-a)^n + \frac{1}{n!}\int_a^b (b-t)^n g^{(n+1)}(t)dt.$$

(3) (Consistency II) Deduce from the previous item that for a constant C independent of τ, T and u we have

$$\begin{aligned} &\|\frac{1}{\tau}(u^{n+1}(x) - u^n(x)) - \frac{1}{2}\Delta(u^{n+1}(x) + u^n(x)) - f(t^{n+\frac{1}{2}})\|_{L_2(\Omega)} \\ &\quad \leq C\tau^{\frac{3}{2}} \left(\|\frac{\partial^3}{\partial t^3} u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|_{L_2(t^n, t^{n+1}; L_2(\Omega))} \right). \end{aligned}$$

(4) From (2) and (4), conclude the following estimate for the error $e^n := u(t^n) - U^n$:

$$\|e^N\|^2_{L_2(\Omega)} \leq C\tau^4 \left(\|\frac{\partial^3}{\partial t^3} u\|^2_{L_2(0,T; L_2(\Omega))} + \|\frac{\partial^2}{\partial t^2}\Delta u\|^2_{L_2(0,T; L_2(\Omega))} \right),$$

where C is a constant independent of τ, T and u.