NUMERICAL ANALYSIS QUALIFIER

August, 2023

Problem 1. Let $\Omega = (0,1) \times (0,1)$ be the unit square. Let $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{0\}\times[0,1]\cup[0,1]\times\{0\}$ consists of the left and bottom edges of Ω and $\Gamma_2 = \{1\}\times(0,1]\cup(0,1]\times\{1\}$ consists of the top and right edges of Ω. Consider the PDE:

$$\begin{aligned} -\Delta u + u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \Gamma_1, \\ \frac{\partial u}{\partial n} &= g \text{ on } \Gamma_2. \end{aligned}$$

Here $\frac{\partial}{\partial n}$ is the outward normal directional derivative.

(a) Derive the weak form of the above problem. Be sure to include the definition of an appropriate variational space.
(b) Prove that the weak form you derived has a unique solution.
(c) Assume $\mathcal{T}_h$ is a shape-regular, quasiuniform triangulation of Ω. Using affine Lagrange elements, define a finite element method for this problem.
(d) State and prove an appropriate energy error estimate for the finite element method you defined above, assuming that $u \in H^2(\Omega)$.

You may use without proof the following results as long as you accurately state them:

- An appropriate trace inequality.
- An appropriate Poincaré-type inequality.
- Finite element approximation error estimates.
- Céa's Lemma.

Problem 2. Let $T \subset \mathbb{R}^2$ be a triangle with vertices v_1, v_2, v_3. Given a fixed polynomial degree $k \geq 1$, define the degree-k Lagrange points using barycentric coordinates on T by $z_{ij\ell} = (\frac{i}{k}, \frac{j}{k}, \frac{\ell}{k})$, $0 \leq i, j, \ell \leq k$ and $i + j + \ell = k$. Let $\Sigma = \{\sigma_{ij\ell}\}_{0\leq i,j,\ell\leq k,\ i+j+\ell=k}$ with $\sigma_{ij\ell}(u) = u(z_{ij\ell})$. Let also $\hat{T}$ be the reference element with vertices $(0,0)$, $(1,0)$, and $(0,1)$, and note carefully below instances where you are asked to prove results only for the reference element $\hat{T}$ versus instances where you are asked to prove results for a generic triangle T.

(a) Show that the triple $(T, \mathbb{P}_k, \Sigma)$ is a finite element, where $\mathbb{P}_k$ is the polynomials of degree k or less on T.
(b) Given $u \in C(T)$, define the Lagrange interpolant $I_h u \in \mathbb{P}_k$ of u by $I_h u(z_{ij\ell}) = u(z_{ij\ell})$, $0 \leq i, j, \ell \leq k$ with $i + j + \ell = k$. (Recall that the result you are asked to prove in Part a guarantees that this definition uniquely specifies $I_h u$.) Prove that if $\hat{u} \in H^2(\hat{T})$, then

$$\|I_h \hat{u}\|_{L_\infty(\hat{T})} \leq C\|\hat{u}\|_{H^2(\hat{T})} \tag{2.1}$$

with C possibly depending on the polynomial degree k but independent of $\hat{u}$.
(c) Show that if $\hat{u} \in H^{k+1}(\hat{T})$, then

$$\|\hat{u} - I_h\hat{u}\|_{L_\infty(\hat{T})} \leq C|\hat{u}|_{H^{k+1}(\hat{T})}.$$

Here C is independent of $\hat{u}$ but may differ from the constant in (2.1).
(d) Now assume that a shape regular triangle T is given by $T = \{A\hat{x} : \hat{x} \in \hat{T}\}$, where $A \in \mathbb{R}^{2\times 2}$ satisfies

$$ch^2 \leq |\text{det}A| \leq Ch^2,\ |A_{mn}| \leq Ch,\ |A^{-1}_{mn}| \leq Ch^{-1},\ 1 \leq m, n \leq 2.$$

Prove that if $u \in H^{k+1}(T)$, then

$$\|u - I_h u\|_{L_2(T)} \leq Ch^{k+1}|u|_{H^{k+1}(T)}.$$

Here the constants c, C are independent of essential quantities but may differ at each occurrence.

You may use without proof the following results as long as you accurately state them:

- An appropriate Sobolev inequality.
- The Bramble-Hilbert Lemma.

Problem 3. Let Ω be a bounded domain and $T > 0$ be a given final time. For $f \in C^0([0,T]; L_2(\Omega))$ and $u_0 \in H_0^1(\Omega)$ given, we consider the parabolic problem consisting in finding $u : [0,T] \to H_0^1(\Omega)$ such that

$$\begin{cases} \int_\Omega u_t v + \int_\Omega \nabla u \nabla v = \int_\Omega f v & \text{for } 0 < t \le T \text{ and } v \in H_0^1(\Omega), \\ u(x,0) = u_0(x) & \text{for } x \in \Omega. \end{cases}$$

We assume that the solution u to the above problem is sufficiently smooth.

(a) Define a spatially semidiscrete degree-k Lagrange finite element method approximating $u(t)$.

(b) Let $u_h(t)$ be the finite element solution defined in part a). Prove an appropriate energy (stability) bound for $\|u_h(T)\|_{H_0^1(\Omega)}$.

(c) Assuming sufficient regularity of u and standard finite element approximation properties, prove that

$$\|(u - u_h)(T)\|_{H_0^1(\Omega)} \le C h^k.$$

Your solution should specify how the constant C in this estimate depends on u and f.