NUMERICAL ANALYSIS QUALIFIER

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Name: ____

Problem 1. Let Ω be a polygonal domain in $\mathbf{R}^2$ and assume that $0 \in \Omega$. Let $u \in H^1(\Omega)$ be the solution of

$$a(u, \varphi) \;=\; l(\varphi), \qquad \text{for all } \; \varphi \in H^1(\Omega), \tag{1.1}$$

where the bilinear form $a(.\,,.)$ and, for a given function $f \in L^2(\Omega)$, the right hand side $l(\,.\,)$ are defined as follows,

$$a(v, \varphi) \;:=\; \int_\Omega (\nabla v \cdot \nabla \varphi \,+\, \|x\|^2\, v\, \varphi) dx, \qquad l(\varphi) \;:=\; \int_\Omega f \varphi dx \qquad \text{for all } \; v, \varphi \in H^1(\Omega).$$

Let $\mathcal{T}_h$, $0 < h < 1$, be a familiy of shape regular triangulations of Ω. The elements of these partitions will be denoted by T_h. Set

$$V_h := \{v_h \in H^1(\Omega) \;:\; v_h|_{T_h} \in \mathcal{P}^1, \quad T_h \in \mathcal{T}_h\},$$

where $\mathcal{P}^1$ denotes the space of polynomials on $\mathbf{R}^2$ of degree at most 1.

- **(a)** Show that the bilinear form $a(.\,,.)$ is coercive in $H^1(\Omega)$.
  Hint: Decompose Ω into $\Omega_i := \Omega \cap B_\varepsilon(0)$ and $\Omega_o := \Omega \setminus \Omega_i$, where $B_\varepsilon(0)$ is a disc with radius ε around 0 such that $B_\varepsilon(0) \subset \Omega$. **You can use without proof that**
  $$\|v\|^2_{L^2(\Omega)} \;\le\; C\left\{\|v\|^2_{L^2(\Omega_o)} \;+\; \|\nabla v\|^2_{L^2(\Omega)^2}\right\}$$
  for all $v \in H^1(\Omega)$, for a suitable constant $C \geq 1$ depending on ε but independent of v.
- **(b)** Given that $a(\cdot,\cdot)$ and $l(\cdot)$ are continuous, there exists a unique weak solution $u \in H^1(\Omega)$ of (1.1). Derive the strong form of problem (1.1) assuming that the solution u is smooth.

Now, consider the following finite element ansatz: find $u_h \in V_h$ such that

$$a(u_h, \varphi_h) \;=\; l(\varphi_h), \quad \forall \varphi_h \in V_h. \tag{1.2}$$

- **(c)** State and prove Cea's Lemma for the error of the FE solution in the H^1-norm.
- **(d)** Assuming that the solution u is in $H^2(\Omega)$, derive an estimate for the error $\|u - u_h\|_{H^1(\Omega)}$. Your final estimate should reflect the correct order of convergence with respect to the mesh parameter h. You may use without proof suitable approximation results for the finite element space V_h.

Problem 2. Consider the unit interval $\Omega = (0, 1)$ and the following 1D parabolic problem:

$$\begin{aligned} \partial_t u(x,t) \;-\; \partial_{xx} u(x,t) \;&=\; f(t,x), && \text{for } x \in \Omega,\; t \in (0,T], \\ u(t,0) = u(t,1) \;&=\; 0 && \text{for } t \in (0,T], \\ u(0,x) \;&=\; u_0(x), && \text{for } x \in \Omega. \end{aligned}$$

Here, $f(t,x)$ and $u_0(x)$ are given, smooth functions.

(a) Derive the variational (in space) formulation of the above problem. What is a suitable function space V?

(b) Discretize the variational formulation in time only (Rothe's method) with the backward Euler scheme.

(c) Let now $\{\psi_j\}_{j=1}^{\infty} \subset H^2(\Omega) \cap H_0^1(\Omega)$ be an orthonormal (in $L_2(\Omega)$) eigenbasis of $-\partial_{xx}$ with homogeneous Dirichlet boundary conditions with corresponding eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots$. That is,

$$-\partial_{xx}\psi_j(x) = \lambda_j \psi_j(x),\ x \in (0,1),\ \psi_j(0) = \psi_j(1) = 0,\ \|\psi_j\|_{L_2(0,1)} = 1,\ \int_0^1 \psi_i \psi_j = \delta_{ij}.$$

Now, write the semi discrete solution defined in part (b) as follows:

$$u_k^n = \sum_{j=1}^{\infty} c_j^n\, \psi_j(x),$$

where k is the time-step size, t_n denotes the time point $t_n = k\,n$. Derive the equation following relation for c_j^n:

$$\frac{c_j^{n+1} - c_j^n}{k} \ + \ \lambda_j c_j^{n+1} \ = \ F_j^{n+1},\ j = 0, 1, 2 \ldots$$

where $F_j^{n+1} \ := \ \int_0^1 f(t_{n+1}, x)\psi_j(x)\mathrm{d}x$.

(d) Prove the coefficientwise stability result

$$|c_j^{n+1}| \leq q_j^{n+1}\,|c_j^0| + k \sum_{m=1}^{n+1} q_j^{n+2-m}\,|F_j^m|, \quad \text{where } q_j \ := \ \frac{1}{1 + k\lambda_j}.$$

Using that $\|u_k^n\|_{L_2(0,1)} = (\sum_{j=1}^{\infty}(c_j^n)^2)^{1/2}$, conclude that if $f = 0$, then

$$\|u_k^n\|_{L_2(0,1)} \leq \|u_0\|_{L_2(0,1)}.$$

Problem 3. Let K be a nondegenerate triangle in $\mathbf{R}^2$. Let a_1, a_2, a_3 be the three vertices of K. Let $a_{ij} = a_{ji}$ denote the midpoint of the segment (a_i, a_j), $i, j \in \{1, 2, 3\}$ and $i \neq j$. Let $\mathcal{P}^2$ be the set of the polynomial functions over K of total degree at most 2. Let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3, \sigma_{12}, \sigma_{23}, \sigma_{31}\}$ be the functionals (or degrees of freedom) on $\mathcal{P}^2$ defined as

$$\sigma_i(p) := p(a_i),\ i \in \{1, 2, 3\} \qquad \sigma_{ij}(p) := p(a_i) + p(a_j) - 2p(a_{ij}),\ i, j = 1, 2, 3,\ i \neq j.$$

(a) Show that Σ is a unisolvent set for $\mathcal{P}^2$ (this means that any $p \in \mathcal{P}^2$ is uniquely determined by the values of the above degrees of freedom applied to p).

(b) Compute the "nodal" basis $\{\psi_j\}_{j=1}^{6}$ of $\mathcal{P}^2$ which corresponds to $\{\sigma_1, \ldots, \sigma_{31}\}$.

Hint for part (a) and (b): Use barycentric coordinates.

(c) Given $u \in C^0(K)$, define an interpolation operator I_h by

$$(I_h u)(x) = \sum_{j=1}^{3} \sigma_j(u)\psi_j(x) + \sigma_{12}(u)\psi_4(x) + \sigma_{23}(u)\psi_5(x) + \sigma_{23}(u)\psi_6(x).$$

Show the following:

(i) $I_h u = u$ if $u \in \mathcal{P}^2$.

(ii) There is a constant C independent of u and K such that

$$\|I_h u\|_{L_\infty(K)} \leq C\|u\|_{L_\infty(K)}.$$

(iii) Finally deduce that

$$\|u - I_h u\|_{L_\infty(K)} \leq C \inf_{\chi \in \mathcal{P}^2} \|u - \chi\|_{L_\infty(K)}.$$