NUMERICAL ANALYSIS QUALIFIER

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Problem 1. Let $\mathbb{P}_2$ be the space of polynomials in two variables spanned by $\{1, x_1, x_2, x_1^2, x_1x_2, x_2^2\}$, let $\hat{T}$ be the reference unit triangle, $\hat{\gamma}$ one side of $\hat{T}$, and $\hat{\pi}$ the standard Lagrange interpolant in $\hat{T}$ with values in $\mathbb{P}_2$.

Recall that there exists a constant C only depending on the geometry of $\hat{T}$ such that

$$\forall v \in H^3(\hat{T}),\ \inf_{p\in\mathbb{P}_2} \|v+p\|_{H^3(\hat{T})} \leq C|v|_{H^3(\hat{T})}.$$

1. State a trace theorem relating $L^2(\hat{\gamma})$ and $H^1(\hat{T})$.
2. Prove that there exists a constant $\hat{C}$ only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that
$$\forall \hat{u} \in H^3(\hat{T}),\ \|\hat{u} - \hat{\pi}(\hat{u})\|_{L^2(\hat{\gamma})} \leq \hat{C}|\hat{u}|_{H^3(\hat{T})}.$$
3. Let Ω be a bounded polygon in $\mathbb{R}^2$, $\mathcal{T}_h$ be a triangulation of Ω and
$$X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega})\,;\, \forall T \in \mathcal{T}_h,\ v_h|_T \in \mathbb{P}_2\}.$$
Let T be a triangle of $\mathcal{T}_h$ with diameter h_T and diameter of inscribed disc ϱ_T, and let γ be one side of T. Let F_T be the affine mapping from $\hat{T}$ onto T and let $\pi_{2,h}$ denote the standard Lagrange interpolant on X_h. Prove that there exists a constant C only depending on the geometry of $\hat{T}$ and $\hat{\gamma}$ such that
$$\forall u \in H^3(T),\ \|u - \pi_{2,h}(u)\|_{L^2(\gamma)} \leq C\sigma_T h_T^{2+1/2}|u|_{H^3(T)},$$
where $\sigma_T = h_T/\varrho_T$.

Problem 2. For $f \in L^2(0,\ell)$, consider the following weak formulation: Seek $(u,v) \in \mathbb{V} := H_0^1(0,\ell) \times H_0^1(0,\ell)$ satisfying for all $(\phi,\psi) \in \mathbb{V}$

$$a((u,v);(\phi,\psi)) := \int_0^\ell u'\phi' + \int_0^\ell v'\psi' - \int_0^\ell v\phi = \int_0^\ell f\psi =: L(\psi). \tag{2.1}$$

1. What is the corresponding strong form satisfied by u (eliminate v)?
2. Show that for all $w \in H_0^1(0,\ell)$
$$\left(\int_0^\ell w^2\right)^{1/2} \leq \left(\int_0^\ell |w'|^2\right)^{1/2}.$$
3. Show that $a(\cdot;\cdot)$ coerces the natural norm on $\mathbb{V}$:
$$|||\phi,\psi||| := \left(\|\phi\|^2_{H^1(0,\ell)} + \|\psi\|^2_{H^1(0,\ell)}\right)^{1/2}$$
and explicitly find a coercivity constant.
4. Let $\mathbb{V}_h$ be a finite dimensional subspace of $\mathbb{V}$. Show that there is a unique $(u_h,v_h) \in \mathbb{V}_h$ satisfying for all $(\phi_h,\psi_h) \in \mathbb{V}_h$
$$a((u_h,v_h);(\phi_h,\psi_h)) = L(\psi_h).$$
5. Prove the estimate
$$|||u-u_h, v-v_h||| \leq C_1 \inf_{(\phi_h,\psi_h)\in\mathbb{V}_h} |||u-\phi_h, v-\phi_h|||$$
where C_1 is a constant independent of h (find C_1 explicitly).

6. You may assume that $u, v \in H_0^1(0,\ell) \cap H^2(0,\ell)$. Propose a discrete space $\mathbb{V}_h$ such that

$$|||u - u_h, v - v_h||| \leq C_2 h(\|u\|_{H^2(0,\ell)} + \|v\|_{H^2(0,\ell)})$$

for a constant C_2 independent of h. Justify your suggestion (you can assume the standard interpolation estimates hold).

Problem 3. Let b be a strictly positive constant and consider the problem: find $u(x,t)$ such that

$$\begin{aligned} &\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1,\ 0 < t \\ &u(x,0) = u_0(x), \quad 0 < x < 1, \\ &u(0,t) = u(1,t),\ t > 0 \end{aligned}$$

where u_0 is a smooth periodic function. Let J and N be positive integers, $x_i = ih$ for $i = 0, ..., J$ where $h = 1/J$ and $t_n = n\tau$ for $n \geq 0$ where $\tau = 1/N$. Also denote by u_j^n the approximation of $u(x_j, t_n)$.

Set $u_j^0 = u_0(x_j)$ and define reccursively u_j^n by the following *Lax* scheme

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{\tau b}{2h}(u_{j+1}^n - u_{j-1}^n), \quad j = 0, ..., J,$$

with the convention that $u_{-1}^n = u_{J-1}^n$ and $u_{J+1}^n = u_1^n$. Show that for all $j = 0, ..., J$ and $n \geq 0$

$$\min_i(u_i^0) \leq u_j^n \leq \max_i(u_i^0)$$

provided $\frac{\tau b}{h} \leq 1$.