**APPLIED ANALYSIS/NUMERICAL ANALYSIS QUALIFIER**
**Numerical Analysis Part, 2 hours**
**August 8, 2018**

**Problem 1.** Let $\Omega := (0,1)^2$ and $u \in H^1_\#(\Omega) := \{u \in H^1(\Omega) \ : \ \int_\Omega u = 0\}$ be such that

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv =: L(v), \qquad \forall v \in H^1_\#(\Omega),$$

where $f \in L^2(\Omega)$ is a given function satisfying $\int_\Omega f = 0$. Accept as a fact that there exists a unique weak solution $u \in H^1_\#(\Omega)$.

The goal of this exercise is to analyze a non-conforming finite element method relaxing the vanishing mean value condition.

(1) Consider the approximate problem: Given $0 < \epsilon \leq 1$, seek $u^\epsilon \in H^1(\Omega)$ such that

$$a_\epsilon(u^\epsilon, v) := \int_\Omega \nabla u^\epsilon \cdot \nabla v + \epsilon \int_\Omega u^\epsilon v = L(v), \qquad \forall v \in H^1(\Omega).$$

(i) Show that the above problem has a unique solution; (ii) show that $\int_\Omega u^\epsilon = 0$, i.e. $u^\epsilon \in H^1_\#(\Omega)$; (iii) Show that there exists a constant $C$ independent of $\epsilon$ such that

$$\|\nabla(u - u^\epsilon)\| \leq C\epsilon \|f\|_{L^2(\Omega)}.$$

Hint: You can use (without proof) the following inequality: There exist a constant $c$ such that for all $v \in H^1_\#(\Omega)$ there holds

$$\|v\|_{L^2(\Omega)} \leq c\|\nabla v\|_{L^2(\Omega)}.$$

(2) Let $V_h$ be the finite element space

$$V_h := \{v_h \in C^0(\overline{\Omega}) \ : \ v|_T \in \mathbb{P}^1, \qquad \forall T \in \mathcal{T}_h\},$$

where $\mathcal{T}_h$ is a subdivision of $\Omega$ made of triangles of diameters $h > 0$.
Consider the discrete problem of finding $u^\epsilon_h \in V_h$ such that

$$a_\epsilon(u^\epsilon_h, v_h) = L(v_h), \qquad \forall v_h \in V_h.$$

(i) Show that $u^\epsilon_h$ exists and is unique in $V_h$; (ii) Prove the following error estimate

$$\|\nabla(u^\epsilon - u^\epsilon_h)\|^2_{L^2(\Omega)} + \epsilon\|u^\epsilon - u^\epsilon_h\|^2_{L^2(\Omega)} \leq c\ (h^2 + \epsilon h^4)|u^\epsilon|^2_{H^2(\Omega)},$$

where $c$ is a constant independent of $h$ and $\epsilon$.
Hint: you can use standard interpolation results without proof.

(3) Assume that $|u^\epsilon|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ for a constant $C$ independent of $\epsilon$ and derive an error estimate for $\|\nabla(u - u^\epsilon_h)\|_{L^2(\Omega)}$. What is the optimal choice for $\epsilon$?

**Problem 2.** Let $T$ be the unit triangle in $\mathbb{R}^2$, with vertices $v_1 = (0,0)$, $v_2 = (1,0)$, and $v_3 = (0,1)$ and edges $e_1 = v_1v_2$, $e_2 = v_2v_3$ and $e_3 = v_3v_1$. Let $RT_0 = \{(a + cx, b + cy) : a, b, c \in \mathbb{R}\}$ (so that members of $RT_0$ are vector functions over $T$, and $[\mathbb{P}_0]^2 \subsetneq RT_0 \subsetneq [\mathbb{P}_1]^2$). Finally, let $\sigma_i(\vec{u}) = \int_{e_i} \vec{u} \cdot \vec{n}_i$, where $\vec{n}_i$ is the outward pointing unit normal vector to $T$ on $e_i$, and let $\Sigma = \{\sigma_1, \sigma_2, \sigma_3\}$.

(a) Show that $(T, RT_0, \Sigma)$ is unisolvent.

(b) Find a basis $\{\vec{\varphi}_1, \vec{\varphi}_2, \vec{\varphi}_3\}$ for $RT_0$ that is dual to $\Sigma$, i.e. $\sigma_i(\vec{\varphi}_j) = \delta_{ij}$ with $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise.

(c) Let $(\Pi\vec{u})(x) = \sum_{i=1}^3 \sigma_i(\vec{u})\vec{\varphi}_i(x)$, $x \in T$ and $\vec{u} \in [H^1(T)]^2$. Show that

$$\|\vec{u} - \Pi\vec{u}\|_{[L_2(T)]^2} \leq C|u|_{[H^1(T)]^2}, \ u \in [H^1(T)]^2.$$

*Note:* You may use standard analysis results such as trace and Poincaré inequalities without proof, but specify carefully which inequalities you are using.

**Problem 3.** Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain, $T > 0$ be a given final time and $\mathbf{b}$ be a given smooth vector valued function satisfying

$$\operatorname{div}(\mathbf{b}(x,t)) = 0 \quad (x,t) \in \Omega \times [0,T] \quad \text{and} \quad \mathbf{b}(x,t) = 0 \quad (x,t) \in \partial\Omega \times [0,T].$$

Consider the time-dependent problem

$$\frac{\partial u}{\partial t}(x,t) + \mathbf{b}(x,t) \cdot \nabla u(x,t) = 0, \qquad (x,t) \in \Omega \times (0,T)$$

together with the initial condition $u(x,0) = u_0$, $x \in \Omega$.

Let $\mathcal{T}_h$ be a subdivision of $\Omega$ made of triangles and

$$V_h := \{v_h \in C^0(\overline{\Omega}) \; : \; v_h|_K \in \mathbb{P}_1 \; \forall K \in \mathcal{T}\}.$$

Choose an integer $N \geq 2$, set $k := T/N$ and $t_n := nk$. Let $u_h^0 \in V_h$ be a given approximation of $u_0$. For $1 \leq n \leq N$ define $u_h^n \in V_h$ recursively by the relations

$$\frac{1}{k}\int_\Omega (u_h^n(x) - u_h^{n-1}(x))v_h(x)\; dx + \int_\Omega (\mathbf{b}(x,t_n) \cdot \nabla u_h^n(x))v_h(x)\; dx = 0, \qquad \forall v_h \in V_h.$$

- Prove that given $u_h^n \in V_h$, the above finite dimensional system has a unique solution $u_h^{n+1} \in V_h$.
- Prove that for $1 \leq n \leq N$

$$\|u_h^n\|_{L^2(\Omega)} \leq \|u_h^0\|_{L^2(\Omega)}.$$

- Is the matrix representing the finite dimensional system symmetric ? Justify your answer.