Applied Mathematics Qualifying Exam

January 12, 2007

Instructions: *Do any four of problems 1 through 5 in part 1, and any three of problems 6 through 9 in part 2.. Show all of your work clearly. Please indicate of the problems you are skipping.*

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem in such a way that it becomes trivial.

Part 1. Do any four problems.

1. Let $\mathcal{H}$ be a complex Hilbert space, with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ being the inner product and norm.
   (a) Define these terms (all for $\mathcal{H}$): bounded linear operator, adjoint of a bounded linear operator, compact linear operator.
   (b) Let $\lambda \in \mathbb{C}$ be fixed. If $K : \mathcal{H} \to \mathcal{H}$ is a compact linear operator, show that the range of the operator $L = I - \lambda K$ is closed.
   (c) Determine the values of λ for which $u = f + \lambda K u$ has a solution for all $f \in \mathcal{H}$, given that $Ku(x) = \int_0^\pi \cos(x-t)u(t)dt$.
2. Let $\mathcal{H}$ and K be as in problem 1. If $\{P_n\}$ is a sequence of orthogonal projections on $\mathcal{H}$ such that $\lim_{n\to\infty}\|P_n u - u\| = 0$ for all $u \in \mathcal{H}$, then show that $\lim_{n\to\infty}\|P_n K - K\|_{op} = 0$. Explain how this is used in the Galerkin method for solving $u - Ku = f$, assuming $I - K$ is invertible.
3. State the Weierstrass Approximation Theorem and sketch a proof of it.
4. State the Courant-Fischer minimax principle. Use it to show that λ_2, the middle eigenvalue of A below, satisfies $\lambda_2 \le -1/3$.

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix},$$

5. Suppose that $Lu := -u'' + e^x u$, with domain $D = \{u \in L^2[0,1] \,|\, u'' \in L^2[0,1],\ u(0) = 0,\ u(1) = 0\}$.

   (a) Show that L is self-adjoint on D. (That is, $L^* = L$ in the inner product of $L^2[0,1]$.)

   (b) Let $H = \{u \in L^2([0,1] \,|\, u' \in L^2[0,1]\}$, together with an inner product $\langle u, v\rangle := \int_0^1 (u'v' + e^x uv)\, dx$ and norm $\|u\| = \sqrt{\langle u, u\rangle}$. Given that H is a Hilbert space, show that $Lu = f$, $f \in L^2[0,1]$ has a unique weak solution.

   (c) Let $S^h(1,0)$ be the space of linear splines on $[0,1]$, with $h = 1/n$. Explain how to use the Rayleigh-Ritz method and the finite element space to approximately solve $Lu = f$.

Part 2. Do any three problems.

6. A mass m is constrained to move on the surface of a unit sphere, subject to a potential $U = U(\theta, \varphi)$. The angles θ and φ are the colatitude and longitude, repectively. The Lagrangian for this system is $L = T - U$, where $T = \frac{m}{2}(\dot{\theta}^2 + \sin^2(\theta)\dot{\varphi}^2)$.

   (a) State Hamilton's principle and use it to write out the equation of motion for the mass.

   (b) Find the momenta p_θ and p_φ conjugate to θ and φ, respectively, and also the Hamiltonian $H(\theta, \varphi, p_\theta, p_\varphi)$ for the system.

   (c) Write down Hamilton's equations for the system. Use them to show that H is a constant of the motion.

7. Consider the Γ-function, which is defined by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. Use the method of Laplace to establish Stirling's formula,

$$\Gamma(x+1) = \sqrt{2\pi}x^{x+1/2}e^{-x}(1 + \mathcal{O}(x^{-1})),\ x \to \infty.$$

8. Let $\mathcal{S}$ be Schwartz space and $\mathcal{S}'$ be the space of tempered distributions.

   (a) Define $\mathcal{S}$ and give the semi-norm topology for it. In addition, define $\mathcal{S}'$.

   (b) Explain how one defines the Fourier transform of a tempered distribution.

   (c) Show that if $T \in \mathcal{S}'$, then $\widehat{T^{(k)}} = (-i\omega)^k \widehat{T}$, where $k = 1, 2, \ldots$. (Use the Fourier transform convention $\hat{f}(\omega) = \int_{\mathbb{R}} f(t)e^{i\omega t}dt$.)

   (d) Let $T(t) = (1 - |t|)_+$, where $(t)_+ = \frac{1}{2}(t + |t|)$. Show that $T''(t) = \delta(t+1) - 2\delta(t) + \delta(t-1)$. Use (8c) to find $\widehat{T}$.

9. Consider the operator $Lu = -u''$ defined on functions in $L^2[0, \infty)$ having u'' in $L^2[0, \infty)$ and satisfying the boundary condition that $u'(0) = 0$; that is, L has the domain

$$\mathcal{D}_L = \{u \in L^2[0, \infty) \,|\, u'' \in L^2[0, \infty) \text{ and } u'(0) = 0\}.$$

   (a) Find the Green's function $G(x, \xi; z)$ for $-G'' - zG = \delta(x - \xi)$, with $G_x(0, \xi; z) = 0$. (This is the kernel for the resolvent $(L - zI)^{-1}$.)

   (b) Employ the spectral theorem to obtain the cosine transform formulas,

$$F(\mu) = \frac{2}{\pi}\int_0^\infty f(x)\cos(\mu x)dx \text{ and } f(x) = \int_0^\infty F(\mu)\cos(\mu x)d\mu\,.$$