Applied Analysis Qualifying Exam

May 22, 2007

Instructions: *Do any 7 of the 9 problems in this exam. Show all of your work clearly. Please indicate which 2 of the 9 problems you are skipping.*

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem in such a way that it becomes trivial.

1. State and prove one of the following theorems:

   (a) The Weierstrass approximation theorem (sketch of proof suffices).

   (b) The Hilbert space projection (decomposition) theorem.

   (c) The Shannon sampling theorem.

2. Suppose that $f(\theta)$ is 2π-periodic function in $C^{(1)}(\mathbb{R})$, and that f'' is piecewise continuous and 2π-periodic. Let c_k denote the k^{th} (complex) Fourier coefficient for f, and let $f_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n^{th} partial sum of the Fourier series for f, $n \geq 1$.

   (a) For $k \neq 0$, show that the Fourier coefficient c_k satisfies the bound

   $$|c_k| \leq \frac{1}{2\pi |k|^2} \|f''\|_{L^1[0,2\pi]}.$$

   (b) Show that both of these hold for f. (The constants are independent of f and n.)

   $$\|f - f_n\|_{L^2[0,2\pi]} \leq C_1 \frac{\|f''\|_{L^1[0,2\pi]}}{\sqrt{n^3}} \text{ and } \|f - f_n\|_{C[0,2\pi]} \leq C_2 \frac{\|f''\|_{L^1[0,2\pi]}}{n}.$$

3. Consider the integral operator $Ku = \int_a^b k(x,\xi) u(\xi) d\xi$.

   (a) Sketch a proof of this: *If K is a Hilbert-Schmidt operator, then K is compact.*

(b) Let $Ku = \int_0^\pi k(x,\xi)u(\xi)d\xi$, where

$$k(x,\xi) = \begin{cases} x-\pi & 0 \le \xi \le x \le \pi, \\ \xi - \pi & 0 \le x < \xi \le \pi. \end{cases}$$

Explain why this K is compact. Show that it is self adjoint and find the eigenvalues and eigenfunctions for K. (Hint: convert the integral equation into a differential equation plus boundary conditions.)

(c) With K as in part (b), for what values of λ will $u = f + \lambda K u$ have a solution for all $f \in L^2[a,b]$? Why?

4. Let $\mathcal{D}$ be the set of compactly supported C^∞ functions defined on $\mathbb{R}$ and let $\mathcal{D}'$ be the corresponding set of distributions.

(a) Define convergence in $\mathcal{D}$ and in $\mathcal{D}'$.

(b) Show that every $\psi \in \mathcal{D}$ satisfies $\psi(x) = (x^2\varphi(x))'$ for some $\varphi \in \mathcal{D}$ if and only if

$$\int_{-\infty}^{\infty} \psi(x)dx = \int_0^\infty \psi(x)dx = \psi(0) = 0.$$

(c) Use the result above to find all $t \in \mathcal{D}'$ that solve $x^2 t' = 0$, in the sense of distributions.

5. A mass m is subject to a force due to a radial potential $V = V(r)$, where r is the radius in spherical coordinates. The angles θ and φ are the colatitude and longitude, repectively.

(a) Find the system's Lagrangian in spherical coordinates.

(b) Find the momenta p_r, p_θ and p_φ conjugate to r, θ and φ, respectively, and also the Hamiltonian $H(r,\theta,\varphi,p_r,p_\theta,p_\varphi)$ for the system.

(c) Write down Hamilton's equations for the system. Use them to show that H is a constant of the motion.

6. Use Laplace's method and Watson's lemma to find the first two terms of an asymptotic expansion for

$$I(x) = \int_0^\infty e^{-x\cosh(t)} \sinh^{1/2}(t)dt, \ x \to +\infty.$$

7. Let $\sigma \geq 0$ and consider the Sturm-Liouville problem $(xu')' + \lambda x u = 0$, with $u(0)$ bounded and $u'(1) + \sigma u(1) = 0$.

   (a) Show that this S-L problem has the solution $u = J_0(\sqrt{\lambda}x)$, where J_0 is the 0 order Bessel function, and where the eigenvalues must satisfy $\sigma J_0(\sqrt{\lambda}) + \sqrt{\lambda} J_0'(\sqrt{\lambda}) = 0$.

   (b) Write out the functional that must be minimized by u, subject to the constraint $H(u) = \int_0^1 u^2(x) x dx = 1$, to get the S-L problem and the boundary conditions.

   (c) Use the Courant-Fischer minimax principle to determine how the k^{th} eigenvalue $\lambda_k(\sigma)$ behaves as σ increases from 0.

8. Consider the Schrödinger operator with a δ-function potential, $Hu = -u'' + \alpha\delta(x)u$, where $\alpha > 0$. For a plane wave incoming from $-\infty$, find the reflection and transmission coefficients.

9. Let $Lu = -x(xu')'$ be defined on functions satisfying the boundary condition that $u(0) = 0$, and let $\mathcal{H}$ be the weighted L^2 space, with the inner product $\langle f, g \rangle = \int_0^\infty f(x)\overline{g(x)}\frac{dx}{x}$. You are given that L will be self adjoint if its domain is

$$\mathcal{D}_L = \{u \in \mathcal{H} \mid Lu \in \mathcal{H} \text{ and } u(0) = 0\}.$$

   (a) Find the Green's function $G(x, \xi; z)$ for $-x(xG')' - zG = \delta(x - \xi)$, with $G(0, \xi; z) = 0$, $G(x, \xi; z) \in L^2[0, \infty)$. (This is the kernel for the resolvent $(L - zI)^{-1}$.)

   (b) Employ the spectral theorem (and Stone's formula) to obtain the Mellin transform formulas,

$$F(s) = \int_0^\infty x^{s-1} f(x) dx \text{ and } f(x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} x^{-s} F(s) ds\,.$$