Applied Analysis Qualifying Exam

May 21, 2008

Instructions: *Do any 7 of the 9 problems in this exam. Show all of your work clearly. Please indicate which 2 of the 9 problems you are skipping.*

Policy on misprints: The qualifying exam committee tries to proofread exams as carefully as possible. Nevertheless, the exam may contain a few misprints. If you are convinced a problem has been stated incorrectly, indicate your interpretation in writing your answer. In such cases, do *not* interpret the problem in such a way that it becomes trivial.

1. Consider the function $I(x) := \int_0^1 t^x(2-t)^x dt$. Use Laplace's method and Watson's lemma to find the asymptotic expansion $I(x)$ to order $\mathcal{O}(x^{-2})$, for $x \to \infty$.

2. Let $\mathcal{S}$ be Schwartz space, $\mathcal{S}'$ be the space of tempered distributions, and let $T(x) = (x)_+ - 3(x-2)_+ + 2(x-3)_+$, where $(x)_+ = \begin{cases} x & 0 \le x \\ 0 & x < 0 \end{cases}$.

   (a) Define $\mathcal{S}$ and give the semi-norm topology for it. In addition, define $\mathcal{S}'$.

   (b) Find the distributional second derivative, T'', and the Fourier transform of T''. Use these results to find the Fourier transform of T.

3. Consider a functional $J[u]$, where $u \in V$, and V is a Banach space.

   (a) Define the Frechét derivative and the Gâteaux derivative for $J[u]$. Illustrate the difference between them with a simple two variable example.

   (b) Consider the constrained functional,

   $$J[u] = \int_0^1 pu'^{\,2} dx + \sigma u(1)^2, \;\; H[u] = \int_0^1 u^2 dx = 1,$$

   where $u \in C^{(1)}[0,1]$, $u(0) = 0$, and $\sigma > 0$. Calculate the variational derivative of the problem, using Lagrange multipliers. Find the Sturm-Liouville eigenvalue problem associated with it.

(c) How does the second eigenvalue of this problem compare with the second eigenvalue of the corresponding Dirichlet problem, i.e., $u(0) = u(1) = 0$? Explain your answer.

4. Let $\mathcal{H}$ be a Hilbert space with inner product and norm given by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. You may assume that $\mathcal{H}$ is a real Hilbert space.

   (a) State and prove the Riesz Representation Theorem.

   (b) Suppose that $\mathcal{H} \subset C[0,1]$ and that, for $f \in \mathcal{H}$, $\|f\|_{C[0,1]} \leq \|f\|$. Show that for every $\xi \in [0,1]$ there is a function $K_\xi(\cdot) \in \mathcal{H}$ for which $f(\xi) = \langle f, K_\xi \rangle$.

   (c) Consider N distinct points $0 \leq \xi_1 < \xi_2 < \cdots < \xi_N \leq 1$ and let $\mathcal{U} := \text{span}\{K_{\xi_j}\}_{j=1}^N$, which is a finite dimensional subspace of $\mathcal{H}$. Show that for any $f \in \mathcal{H}$, the orthogonal projection of f onto $\mathcal{U}$, $\tilde{f} = \text{Proj}_{\mathcal{U}} f$, satisfies $\tilde{f}(\xi_j) = f(\xi_j)$, $j = 1, \ldots, N$.

5. The degree n Chebyshev polynomial can be defined via the Rodrigues' formula,

$$T_n(x) = (-2)^n \frac{n!}{(2n)!} (1-x^2)^{1/2} \frac{d^n}{dx^n} \left([1-x^2]^{n-1/2} \right).$$

   (a) Using the Rodrigues' formula, show that, in the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) (1-x^2)^{-1/2} dx$, T_n is orthogonal to all polynomials of degree $n-1$ or less.

   (b) Show that the generating function for the Chebyshev polynomials is

$$\Phi(x, w) := \sum_{n=0}^{\infty} T_n(x) w^n = \frac{1 - xw}{1 - 2xw + w^2}.$$

6. Do *one* of the following.

   (a) State the Weierstrass approximation theorem and sketch a proof.

   (b) Let $\mathcal{H}$ be a complex, separable Hilbert space with inner product and norm given by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. If L is a self-adjoint operator defined on a domain $D \subseteq \mathcal{H}$, show that L has no residual spectrum and that its spectrum is real.

7. Consider the Schrödinger operator $Hu = -u'' + q(x)u$, with a compactly supported, continuous potential $q(x) \geq 0$. Show that the left and right transmission coefficients are equal; that is, $T_L(k) = T_R(k)$.

8. Let L to be the self-adjoint operator $Lu = -u''$, where $D(L) = \{u \in L^2(\mathbb{R}) : u'' \in L^2(\mathbb{R})\}$.

   (a) Find the Green's function for L.

   (b) Employ Stone's formula (i.e., the spectral theorem for self-adjoint operators) to obtain the Fourier transform,

$$F(\mu) = \int_{-\infty}^{\infty} f(x)e^{i\mu x}dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\mu)e^{-i\mu x}d\mu.$$

9. Let $\mathcal{H}$ be a complex (separable) Hilbert space, with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ being the inner product and norm.

   (a) Let $\lambda \in \mathbb{C}$ be fixed. If $K : \mathcal{H} \to \mathcal{H}$ is a compact linear operator, show that the range of the operator $L = I - \lambda K$ is closed.

   (b) Briefly explain why the operator $Ku(x) = \int_0^\pi \sin(x-t)u(t)dt$ is compact on $L^2[0,\pi]$.

   (c) Determine the values of λ for which $u = f + \lambda Ku$ has a solution for all $f \in \mathcal{H}$, given that K is the operator in 9b.